Truth and Envy with Capacitated Valuations

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Abstract

We study auctions with additive valuations where agents have a limit on the number of goods they may receive. We refer to such valuations as *capacitated* and seek mechanisms that maximize social welfare and are simultaneously incentive compatible, envy-free, individually rational, and have no positive transfers.

If capacities are infinite, then sequentially repeating the 2nd price Vickrey auction meets these requirements. In 1983, Leonard showed that for unit capacities, VCG with Clarke Pivot payments is also envy free. For capacities that are all unit or all infinite, the mechanism produces a Walrasian pricing (subject to capacity constraints).

Here, we consider general capacities. For homogeneous capacities (all capacities equal) we show that VCG with Clarke Pivot payments is envy free (VCG with Clarke Pivot payments is always incentive compatible, individually rational, and has no positive transfers). Contrariwise, there is no incentive compatible Walrasian pricing.

For heterogeneous capacities, we show that there is no mechanism with all 4 properties, but at least in some cases, one can achieve both incentive compatibility and envy freeness.

1 Introduction

We consider settings where a set $[s] = \{1, \ldots, s\}$ of s goods should be allocated amongst n agents with private valuations. An agent's valuation function is a mapping from every subset of the goods into the non negative reals. A *mechanism* receives the valuations of the agents as input, and determines an allocation a_i and a payment p_i for every agent. We assume that agents have quasi-linear utilities; that is, the utility of agent i is the difference between her valuation for the bundle allocated to her and her payment.

We seek mechanisms that are

1. Efficient — the mechanism maximizes the sum of the valuations of the agent. Alternately, efficient mechanisms are said to maximize social welfare.

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2. Incentive compatible (truthful) — it is a dominant strategy for agents to report their private information [8].

3. Envy free - no agent wishes to exchange her outcome with that of another [5, 6, 16, 10, 11, 18].

4. Make no positive transfers — the payments of all agents are non-negative.

5. Individually rational for agents — no agent gets negative utility.

We use the acronyms IC, EF, NPT, and IR to denote incentive compatible, envy-free, no positive transfers, and individually rational, respectively.

Our main results concerns the class of capacitated valuations: every agent i has an associated capacity c_i, and her value is additive up to the capacity, *i.e.*, for every set $S \subseteq [s]$,

$$v_i(S) = \max\left\{ \sum_{j \in T} v_i(j) \middle| T \subseteq S, |T| = c_i \right\},$$

where $v_i(j)$ denotes the agent i's valuation for good j.

Consider the following classes of valuation functions:

1. Gross substitutes: good x is said to be a gross substitute of good y if the demand for x is monotonically non-decreasing with the price of y, *i.e.*,

$$\partial(\text{demand } x)/\partial(\text{price } y) \geq 0 \ .$$

   A valuation function is said to obey the gross substitutes condition if for every pair of goods x and y, good x is a gross substitute of good y.

2. Subadditive valuations: A valuation $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ is said to be subadditive if for every two disjoint subsets $S, T \subseteq [s]$, $v(S) + v(T) \geq v(S \cup T)$.

3. Superadditive valuations: A valuation $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ is said to be superadditive if for every two disjoint subsets $S, T \subseteq [s]$, $v(S) + v(T) \leq v(S \cup T)$.

Capacitated valuations are a subset of gross substitutes, which are themselves a subset of subadditive valuations.

In a Walrasian equilibrium (See [7]), prices are *item prices*, that is, prices are assigned to *individual goods* so that every agent chooses a bundle that maximizes her utility and the market clears. Thus, Walrasian prices automatically lead to an envy free allocation. Every Walrasian pricing gives a mechanism that is efficient and envy free, has no positive transfers, and is individually rational [2].

We remark that while Walrasian pricing $\Rightarrow$ EF, NPT, IR, the converse is not true. Even a mechanism that is EF, NPT, IR, *and* IC does not imply Walrasian prices. Note that envy free prices may be assigned to bundles of goods which cannot necessarily be interpreted as item prices. It is well known that in many economic settings, bundle prices are more powerful than item prices [1, 14, 3].

Gul and Stacchetti [7] showed that every allocation problem with valuations satisfying gross substitutes admits a Walrasian equilibrium. For the superset of gross substitutes, subadditive valuations, a Walrasian equilibrium may not exist.

As capacitated valuations are also gross substitutes (see Theorem 2.4 in Section 2.2), it follows that capacitated valuations always have a Walrasian equilibrium. Walrasian prices, however, may not be incentive compatible. In fact, we show (Proposition 3.1) that even with 2 agents with capacities 2 and 3 goods, there is no IC mechanism that produces a Walrasian equilibrium.

For superadditive valuations it is known that Walrasian equilibrium may not exist. Pápai [13] has characterized the family of mechanisms that are simultaneously EF and IC under superadditive valuations. In particular, VCG with Clarke pivot payments satisfies these conditions. However, Pápai's result for superadditive valuations does not hold for subadditive valuations. Moreover, Clarke pivot payments do not satisfy envy freeness even for the more restricted family of capacitated valuations, as demonstrated in the following example:

Example 1.1. Consider an allocation problem with two agents, $\{1, 2\}$, and two goods, $\{a, b\}$. Agent 1 has capacity $c_1 = 1$ and valuation $v_1(a) = v_1(b) = 2$, and agent 2 has capacity $c_2 = 2$ and valuation $v_2(a) = 1$, $v_2(b) = 2$. According to VCG with Clarke pivot payments, agent 1 is given a and pays 1, while agent 2 is given b and pays nothing (as he imposes no externality on agent 1). Agent 1 would rather switch with agent 2's allocation and payment (in which case, her utility grows by 1), therefore, the mechanism is not envy free.

Two extremal cases of capacitated valuations are "no capacity constraints", or, all capacities are equal to one. If capacities are infinite, running a Vickrey 2nd price auction [17] for every good, independently, meets all requirements (IC + Walrasian $\Rightarrow$ efficient, IC, EF, NPT, IR). If all agent capacities are one, [9] shows that VCG with Clarke pivot payments is envy free, and it is easy to see that it also meets the stronger notion of an incentive compatible Walrasian equilibrium. For arbitrary capacities (not only all ∞ or all ones), we distinguish between *homogeneous* capacities, where all agent capacities are equal, and *heterogeneous* capacities, where agent capacities are arbitrary.

When considering incentive compatible and heterogeneous capacities, we distinguish between capacitated valuations with *public* or *private* capacities: being incentive compatible with respect to private capacities and valuation is a more difficult task than incentive compatible with respect to valuation, where capacities are public. In this paper, we primarily consider public capacities.

The main results of this paper (which are also summarized in Figure 1) are as follows:

- For arbitrary homogeneous capacities c, such that
  $(c \equiv c_1 = c_2 = \cdots = c_n)$:
  - VCG with Clarke pivot payments is efficient, IC, NPT, IR, and EF. (Section 3).
  - However, there is no incentive compatible mechanism that produces Walrasian prices, even for $c = 2$. (Section 3).
- For arbitrary heterogeneous capacities
  $c = (c_1, c_2, \ldots, c_n)$:
  - Under the VCG mechanism with Clarke Pivot payments (public capacities), a higher capacity agent will never envy a lower capacity agent. (Section 3).
  - There is no mechanism that is IC, NPT, and EF (for public and hence also for private capacities). (Section 4).
  - We also deal with some special cases:

	Subadditive	Gross substitutes	capacitated - heterogeneous	capacitated - homogeneous
Walras.	NO [7]	YES [7]	($\rightarrow$) YES	($\rightarrow$) YES
Walras.+IC	NO ($\leftarrow$)	NO ($\leftarrow$)	NO ($\leftarrow$)	**NO** (Proposition 3.1)
EF + IC	? **YES** for $m = 2, n = 2$ (Corollary 6.2)	? ($\rightarrow$)**YES** for $m = 2, n = 2$	? **YES** for $m = 2$ (Prop. 5.1)	YES ($\uparrow$)
EF + IC + NPT	NO ($\leftarrow$)	NO ($\leftarrow$)	**NO** (Theorem 4.1)	**YES** (Corollary 3.10)

Figure 1: This table specifies the existence of a particular type of mechanism (rows) for various families of valuation functions (columns). Efficiency is required in all entries. The valuation families satisfy capacitated homogeneous $\subset$ capacitated heterogeneous $\subset$ gross substitutes $\subset$ subadditive. Wherever results are implied from other table entries, this is specified with corresponding arrows. We note that for the family of additive valuations (no capacities), all entries are positive, as the Clarke pivot mechanism satisfies all properties.

- 2 agents, public capacities - there exist mechanisms that are IC, IR, and EF. (Section 5).
- 2 agents, 2 goods - there exist mechanisms that are IC, IR, and EF for every subadditive valuation. (Section 6).

2 Model and Preliminaries

Let $[s] = \{1, \ldots, s\}$ be a set of goods to be allocated to a set $[n] = \{1, \ldots, n\}$ of agents.

An allocation $a = (a_1, a_2, \ldots, a_n)$ assigns agent i the bundle $a_i \subseteq [s]$ and is such that $\bigcup_i a_i \subseteq [s]$ and $a_i \cap a_j = \emptyset$ for $i \neq j$.[1] We use $\mathcal{L}$ to denote the set of all possible allocations.

For $S \subseteq [s]$, let $v_i(S)$ be the valuation of agent i for set S. Let $v = (v_1, v_2, \ldots, v_n)$, where v_i is the valuation function for agent i.

Let V_i be the domain of all valuation functions for agent $i \in [n]$, and let $V = V_1 \times V_2 \times \cdots \times V_n$.

An allocation function $a : V$ maps $v \in V$ into an allocation

$$a(v) = (a_1(v), a_2(v), \ldots, a_n(v)) \ .$$

A payment function $p : V$ maps $v \in V$ to $\mathbb{R}^n_{\geq 0}$: $p(v) = (p_1(v), p_2(v), \ldots, p_n(v))$, where $p_i(v) \in \mathbb{R}_{\geq 0}$ is the payment of agent i. Payments are from the agent to the mechanism (if the payment is negative then this means that the transfer is from the mechanism to the agent).

A mechanism is a pair of functions, $M = \langle a, p \rangle$, where a is an allocation function, and p is a payment function. For a valuation v, the utility to agent i in a mechanism $\langle a, p \rangle$ is defined as $v_i(a_i(v)) - p_i(v)$. Such a utility function is known as quasi-linear.

For a valuation v, we define (v_i', v_{-i}) to be the valuation obtained by substituting v_i by v_i', i.e.,

$$(v_i', v_{-i}) = (v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_n).$$

[1] Here we deal with indivisible goods, although our results also extend to divisible goods with appropriate modifications.

A mechanism is incentive compatible (IC) if for all i, v, and v_i':

$$v_i(a_i(v)) - p_i(v) \geq v_i(a_i(v_i', v_{-i})) - p_i(v_i', v_{-i});$$

this holds if and only if

$$p_i(v) \leq p_i(v', v_{-i}) + \Big(v_i(a_i(v)) - v_i(a_i(v_i', v_{-i}))\Big). \tag{1}$$

A mechanism is envy free (EF) if for all $i, j \in [n]$ and all v:

$$v_i(a_i(v)) - p_i(v) \geq v_i(a_j(v)) - p_j(v);$$

this holds if and only if

$$p_i(v) \leq p_j(v) + \Big(v_i(a_i(v)) - v_i(a_j(v))\Big). \tag{2}$$

Given valuation functions $v = (v_1, v_2, \ldots, v_n)$, a social optimum Opt is an allocation that maximizes the sum of valuations

$$\mathrm{Opt} \in \arg\max_{a \in \mathcal{L}} \sum_{i=1}^{n} v_i(a_i) \ .$$

Likewise, the social optimum when agent i is missing, Opt^{-i}, is the allocation

$$\mathrm{Opt}^{-\mathrm{i}} \in \arg\max_{a \in \mathcal{L}} \sum_{j \in [n] \setminus \{i\}} v_j(a_j) \ .$$

2.1 VCG mechanisms

A mechanism $M = \langle a, p \rangle$ is called a VCG mechanism [17, 4] if:

- $a(v) = \mathrm{Opt}$, and
- $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j(a_j(v))$, where h_i does not depend on v_i, $i \in [n]$.

For connected domains, the only efficient incentive compatible mechanism is VCG (See Theorem 9.37 in [12]). Since capacitated valuations induce a connected domain, we get the following proposition.

Proposition 2.1. With capacitated valuations, a mechanism is efficient and IC if and only if it is VCG.

VCG with *Clarke pivot payments* has

$$h_i(v_{-i}) = \max_{a \in \mathcal{L}} \sum_{j \neq i} v_j(a) \qquad (= \sum_{j \neq i} v_j(\mathrm{Opt}_j^{-i})).$$

Agent valuations for bundles of goods are non negative. The only mechanism that is efficient, incentive compatible, individually rational, and with no positive transfers is VCG with Clarke pivot payments.

The following proposition, which appears in [13], provides a criterion for the envy freeness of a VCG mechanism.

Proposition 2.2. [13] Given a VCG mechanism, specified by functions $\{h_i\}_{i \in [n]}$, agent i does not envy agent j iff for every v,

$$h_i(v_{-i}) - h_j(v_{-j}) \leq v_j(\mathrm{Opt}_j) - v_i(\mathrm{Opt}_j).$$

2.2 Gross substitutes and capacitated valuations

We define the notion of gross substitute valuations and show that every capacitated valuation (i.e., additive up to the capacity) has the gross substitutes property. As this discussion refers to a valuation function of a single agent, we omit the index of the agent.

Fix an agent and let $D(p)$ be the collection of all sets of goods that maximize utility for the agent under price vector p, $D(p) = \arg\max_{S \subseteq [s]}\{v(S) - \sum_{j \in S} p_j\}$.

Definition 2.3. [7] A valuation function $v : 2^{[s]} \to \mathbb{R}_{\geq 0}$ satisfies the gross substitutes condition if the following holds: Let $p = (p_1, \ldots, p_s)$ and $q = (q_1, \ldots, q_s)$ be two price vectors such that the price for good j is no less under q than under p: *i.e.*, $q_j \geq p_j$, for all j. Consider the set of all items whose price is the same under p and q, $E(p, q) = \{1 \leq j \leq s | p_j = q_j\}$, then for any $S^p \in D(p)$ there exists some $S^q \in D(q)$ such that $S^p \cap E(p, q) \subseteq S^q \cap E(p, q)$.

Theorem 2.4. Every capacitated valuation function (additive up to the capacity) obeys the gross substitutes condition.

Proof. Fix a capacitated valuation v, and prices p, q such that $p_i \leq q_i$ for every good i. Fix also some set $S^p \in D(p)$. Let $S^{pq} = \{i \in S^p : p_i = q_i\}$ $(= S^p \cap E(p, q))$. We show that there exists a set in $D(q)$ that contains the set S^{pq}.

Let S^q be an arbitrary set in $D(q)$. Consider the following case analysis:

1. $S^{pq} \subseteq S^q$: we're done.

2. $S^q \subset S^p$ but $S^{pq} \nsubseteq S^q$, this means that S^q is smaller than the capacity of the agent. Let $\tilde{S}^q = S^q \cup (S^{pq} \setminus S^q)$, it follows from the optimality of S^p that for all goods $j \in S^{pq}$ $(\subseteq S^p)$, $v(j) - p_j \geq 0$. Thus,the utility from $\tilde{S}^q$ is at least equal to the utility from S^q, ergo, $\tilde{S}^q \in D(q)$.

3. $S^q \nsubseteq S^p$ and $S^{pq} \nsubseteq S^q$, : Let $x = \min\{|S^q \setminus S^p|, |S^{pq} \setminus S^q|\}$. Replace x arbitrary goods in $(S^q \setminus S^p)$ by x arbitrary goods in $(S^{pq} \setminus S^q)$, and let $\tilde{S}^q$ denote the result.

   From the optimality of S^p, for every $i \in (S^{pq} \setminus S^q)$ and $j \in (S^q \setminus S^p)$ we have $v(j) - p_j \leq v(i) - p_i$. Substituting $p_j \leq q_j$ and $p_i = q_i$, we get $v(j) - q_j \leq v(i) - q_i$, and therefore the utility under q, $u^q(S^q \cup \{i\} \setminus \{j\}) \geq u^q(S^q)$.

   Inductively, it holds that $u^q(\tilde{S}^q) \geq u^q(S^q)$, and thus $\tilde{S}^q \in D(q)$. If $S^{pq} \subseteq \tilde{S}^q$ then we're done as in case 1 above. Otherwise, we can apply case 2 above.

$\square$

As a corollary, we get that capacitated valuations admit a Walrasian equilibrium. However, not necessarily within an IC mechanism.

3 Envy Free and Incentive Compatible Assignments with Capacities

The main result of this section is that Clarke pivot payments are envy free when capacities are homogeneous. This follows from a stronger result, which we establish for heterogeneous capacities, showing that with Clarke pivot payments, no agent envies a lower-capacity agent.

We first observe that one cannot aim for an incentive compatible mechanism with Walrasian prices (if this was possible then envy freeness would follow immediately).

	a	b	c
Agent 1	$1+\epsilon$	$1+\epsilon$	$1-\epsilon$
Agent 2	$1-\epsilon/2$	1	$1+\epsilon$

(a) Matrix v

	a	b	c
Agent 1	$1-\epsilon$	0	0
Agent 2	$1-\epsilon/2$	1	$1+\epsilon$

(b) Matrix v'

Figure 2: No IC mechanism with Walrasian pricing for these inputs.

3.1 No Incentive Compatible Walrasian pricing

Proposition 3.1. Capacitated valuations with homogeneous capacities $c \geq 2$ have no incentive compatible mechanism which produces Walrasian prices.

Proof. Consider the valuations v given in Figure 2(a), which represents valuations for three goods and two agents, each with capacity 2. Assume that Walrasian prices exist. *I.e.*, for every valuation matrix v there exist prices $p_a(v)$, $p_b(v)$, and $p_c(v)$ such that every agent chooses a bundle of maximal utility under these prices, and this allocation maximizes social welfare. The social optimum has

$$\begin{aligned} \mathrm{Opt}_1(v) &= \{a, b\} \\ \mathrm{Opt}_2(v) &= \{c\}. \end{aligned}$$

It follows that the price paid by agent 1 for $\{a, b\}$ is

$$p_1(v) = p_a(v) + p_b(v). \tag{3}$$

However, it also follows from Proposition 2.1 that the only mechanism that is efficient and incentive compatible is the VCG mechanism. Therefore, the price paid by agent 1 is also of the following form:

$$\begin{aligned} p_1(v) &= h_1(v_2) - v_2(\mathrm{Opt}_2(v)) \\ &= h_1(v_2) - v_2(\{c\}) = h_1(v_2) - (1+\epsilon), \end{aligned}$$

for some function h_1 (that does not depend on v_1).

Combining Equations (3) and (4) we get that

$$p_a(v) + p_b(v) = h_1(v_2) - (1+\epsilon). \tag{4}$$

If $p_a(v) < 1 - \epsilon/2$ then agent 2 would choose good a (as agents choose bundles of maximal utility under Walrasian pricing). As agent 2 receives $\mathrm{Opt}_2 = \{c\}$ it follows that $p_a(v) \geq 1 - \epsilon/2$. Similarly, $p_b(v) \geq 1$. Substituting in (4) gives

$$h_1(v_2) \geq 3 + \epsilon/2. \tag{5}$$

Now consider the valuations v' given in Figure 2(b). The social optimum here is $\mathrm{Opt}_1(v') = \{a\}$ and $\mathrm{Opt}_2(v') = \{b, c\}$. As the mechanism is VCG, the payment for agent 1 must be of the form $p_1(v') = h_1(v'_2) - v'_2(\mathrm{Opt}_2(v'))$ for h_1 that does not depend on v'_1. As $v'_2 = v_2$, we have

$$\begin{aligned} p_1(v') &= h_1(v'_2) - v'_2(\mathrm{Opt}_2(v')) \\ &= h_1(v_2) - v_2(\{b, c\}) \\ &= h_1(v_2) - 2 - \epsilon \\ &\geq 1 - \epsilon/2. \end{aligned}$$

The last inequality follows from Equation (5).

Under matrix v' agent 1 gets the bundle $\mathrm{Opt}_1(v') = \{a\}$ and hence $p_a(v') = p_1(v') \geq 1 - \epsilon/2$. The value to agent 1 of good a is $1 - \epsilon$. Agent 1 receives $\mathrm{Opt}_1(v') = \{a\}$. The utility to agent 1 is $v'_1(\{a\}) - p_a(v') \leq (1-\epsilon) - (1-\epsilon/2) < 0$ in contradiction to the individual rationality implied by Walrasian pricing. □

3.2 Truthful and Envy free Capacitated Allocations

The following theorem establishes a general result for capacitated valuations: in a VCG mechanism with Clarke-pivot payments, no agent will ever envy a lower-capacity agent.

Theorem 3.2. If we apply the VCG mechanism with Clarke-pivot payments on the assignment problem with capacitated valuations, then

- The mechanism is incentive compatible, individually rational, and makes no positive transfers (follows from VCG with CPP).
- No agent of higher capacity envies an agent of lower or equal capacity.

The input to the VCG mechanism consists of capacities and valuations. The agent capacity, $c_i \geq 0$ (the capacity of agent i), is publicly known. The number of units of good j, $q_j \geq 0$ is also public knowledge. The valuations — $v_i(j)$ — the value to agent i of a unit of good j, are private.

3.2.1 The b-Matching Graph

Given capacities c_i, q_j, and a valuation matrix v, we construct an edge-weighted bipartite graph G as follows:

- We associate a vertex with every agent $i \in [n]$ on the left, let $\mathcal{A}$ be the set of these vertices.
- We associate a vertex with every good $j \in [s]$ on the right, let $\mathcal{I}$ be the set of these vertices.
- Edge (i, j), $i \in \mathcal{A}$, $j \in \mathcal{I}$, has weight $v_i(j)$.
- Vertex $i \in \mathcal{A}$ (associated with agent i) has *degree constraint* c_i.
- Vertex $j \in \mathcal{I}$ (associated with good j) has degree constraint q_j.

We seek an allocation a $(= a(v))$ where a_{ij} is the number of units of good j allocated to agent i. The value of the allocation is $v(a) = \sum_{ij} a_{ij} v_i(j)$. We seek an allocation of maximal value that meets the degree constraints: $\sum_j a_{ij} \leq c_i$, $\sum_i a_{ij} \leq q_j$, this is known as a b-matching problem and has an integral solution if all constraints are integral, see [15]. Let $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ denote the i'th row of a, which corresponds to the bundle allocated to agent i.

Let $v_k(a_i) = \sum_{j \in [s]} a_{ij} v_k(j)$ denote the value to agent k of bundle a_i. Let M denote some allocation that attains the maximal social value, $M \in \arg\max_a v(a)$. Finally, let G^{-i} be the graph derived from G by removing the vertex associated with agent i and all its incident edges, and let M^{-i} be a matching of maximal social value with agent i removed.

Specializing the Clarke-pivot rule to our setting, the payment of agent k is

$$p_k = v(M^{-k}) - v(M) + v_k(M_k) \ . \tag{6}$$

In the special case of permutation games (the number of agents and goods is equal, and every agent can receive at most one good), the social optimum corresponds to a maximum weighted matching in G. Permutation games were first studied by [9] who showed that Clarke-pivot payments are envy free. However, the shadow variables technique used in this proof does not seem to generalize for larger capacities.

Remark: Our proof is given in terms of fractional allocations (where $a_{ij} \geq 0$, $\sum_j a_{ij} \leq c_i$, $\sum_i a_{ij} \leq q_j$) but also holds for integral allocations (where $a_{ij} \in Z_{\geq 0}$, $\sum_j a_{ij} \leq c_i$, $\sum_i a_{ij} \leq q_j$). This is because when capacities and quantities are integral, there is always an integral social optimum.

Proof. Let agent 1 and agent 2 be two arbitrary agents such that $c_1 \geq c_2$. Agent 1 does not envy agent 2 if and only if

$$v_1(M_1) - p_1 \geq v_1(M_2) - p_2$$

By substituting the Clarke pivot payments (6) and rearranging, this is true if and only if

$$v(M^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2). \tag{7}$$

Thus in order to prove the theorem we need to establish (7).

We construct a new allocation D^{-2} on G^{-2} (from the allocations M and M^{-1}) such that

$$v(D^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2)\ . \tag{8}$$

From the optimality of M^{-2}, it must hold that $v(M^{-2}) \geq v(D^{-2})$. Combining this with (8) shall establish (7), as required.

In what follows we make several preparations for the construction of the allocation D^{-2}. Given M and M^{-1}, we construct a directed bipartite graph G_f on $\mathcal{A} \cup \mathcal{I}$ coupled with a flow f as follows. For every pair of vertices $i \in \mathcal{A}$ and $j \in \mathcal{I}$,

- If $M_{ij} - M^{-1}_{ij} > 0$, then G_f includes arc $i \to j$ with flow $f_{i\to j} = M_{ij} - M^{-1}_{ij}$.
- If $M_{ij} - M^{-1}_{ij} < 0$, then G_f includes arc $j \to i$ with flow $f_{j\to i} = M^{-1}_{ij} - M_{ij}$.
- If $M_{ij} = M^{-1}_{ij}$, then G_f contains neither arc $i \to j$ nor arc $j \to i$.

We define the *excess* of a vertex in G_f to be the difference between the amount of flow flowing out of the vertex and the amount of flow flowing into the vertex. I.e., the excess χ_i of a vertex $i \in \mathcal{A}$ in G_f is

$$\chi_i = \sum_{(i\to j)\in G_f} f_{i\to j} - \sum_{(j\to i)\in G_f} f_{j\to i} = \sum_j \big(M_{ij} - M^{-1}_{ij}\big),$$

and the *excess* χ_j of a vertex $j \in \mathcal{I}$ in G_f is

$$\chi_j = \sum_{(j\to i)\in G_f} f_{j\to i} - \sum_{(i\to j)\in G_f} f_{i\to j} = \sum_i \big(M^{-1}_{ij} - M_{ij}\big).$$

Clearly the sum of all excesses is zero.

A vertex is said to be a *source* if its excess is positive, and said to be a *target* if its excess is negative. The aforementioned definitions imply the following observation.

Observation 3.3. To summarize,

$$0 \leq \sum_j M_{ij}^{-1} + |\chi_i| = \sum_j M_{ij} \leq c_i \quad \forall \text{ source } i \in \mathcal{A}. \tag{9}$$

$$0 \leq \sum_j M_{ij} + |\chi_i| = \sum_j M_{ij}^{-1} \leq c_i \quad \forall \text{ target } i \in \mathcal{A}. \tag{10}$$

$$0 \leq \sum_i M_{ij} + |\chi_j| = \sum_i M_{ij}^{-1} \leq q_j \quad \forall \text{ source } j \in \mathcal{I}. \tag{11}$$

$$0 \leq \sum_i M_{ij}^{-1} + |\chi_j| = \sum_i M_{ij} \leq q_j \quad \forall \text{ target } j \in \mathcal{I}. \tag{12}$$

Using the flow decomposition theorem, we can decompose the flow f into simple paths and cycles, where each path connects a source to a target. Associated with each path and cycle T is a positive flow value $f(T) > 0$. Given an arc $x \to y$, $f_{x \to y}$ is obtained by summing up the values $f(T)$ of all paths and cycles T that contain $x \to y$. Notice that $M_{1j}^{-1} = 0$ for all j and therefore $f_{1 \to j} \geq 0$ for all j. It follows that there are no arcs of the form $j \to 1$ in G_f. The following observation can be easily verified.

Observation 3.4. For each path $P = u_1, u_2, \ldots, u_t$ in a flow decomposition of G_f, where u_1 is a source and u_t is a target, it holds that $f(P) \leq \min\{\chi_{u_1}, |\chi_{u_t}|\}$.

We define the *value* of a path or a cycle $T = u_1, u_2, \ldots, u_t$ in G_f, to be

$$v(P) = \sum_{\substack{u_i \in \mathcal{A}, \\ u_{i+1} \in \mathcal{I}}} v_{u_i}(u_{i+1}) - \sum_{\substack{u_i \in \mathcal{I}, \\ u_{i+1} \in \mathcal{A}}} v_{u_{i+1}}(u_i).$$

It is easy to verify that $\sum_T f(T) \cdot v(T) = v(M) - v(M^{-1})$, where we sum over all paths and cycles T in our decomposition.

We will repeatedly do the following procedure: Let M, M^{-1}, f and G_f be as above.

Lemma 3.5. Let $T = u_1, u_2, \ldots, u_t$ be a cycle in G_f or a path in the flow decomposition of G_f, and let ϵ be the minimal flow along any arc of T. We construct an allocation $\widehat{M}$ ($= \widehat{M}(T)$) from M by canceling the flow along T, start with $\widehat{M} = M$ and then for each $(u_i, u_{i+1}) \in T$ set:

$$\begin{aligned} \widehat{M}_{u_i u_{i+1}} &= M_{u_i u_{i+1}} - \epsilon \qquad u_i \in \mathcal{A}, u_{i+1} \in \mathcal{I} \\ \widehat{M}_{u_{i+1} u_i} &= M_{u_{i+1} u_i} + \epsilon \qquad u_i \in \mathcal{I}, u_{i+1} \in \mathcal{A}\ . \end{aligned}$$

Alternatively, we construct $\widehat{M}^{-1}$ ($= \widehat{M}^{-1}(T)$) from M^{-1}, starting from $\widehat{M}^{-1} = M^{-1}$ and then for each $(u_i, u_{i+1}) \in T$ set

$$\begin{aligned} \widehat{M}^{-1}_{u_i u_{i+1}} &= M^{-1}_{u_i u_{i+1}} + \epsilon \qquad u_i \in \mathcal{A}, u_{i+1} \in \mathcal{I}, \\ \widehat{M}^{-1}_{u_{i+1} u_i} &= M^{-1}_{u_{i+1} u_i} - \epsilon \qquad u_i \in \mathcal{I}, u_{i+1} \in \mathcal{A}. \end{aligned}$$

The allocations $\widehat{M}$, $\widehat{M}^{-1}$ are valid (do not violate capacity constraints).

Proof. If T is a cycle then our manipulations do not affect the total quantity allocated to an agent, nor the total demand for a good.

If T is a path, then our manipulations do not affect the quantities/demands for all internal vertices, *i.e.*, it is sufficient to show that the capacities/demands of u_1 and u_t are not exceeded.

It follows from Observation 3.4 that the flow along a path T, $f(T) \leq \min\{\chi_{u_1}, |\chi_{u_t}|\}$.

Consider the first vertex along T, u_1, if u_1 is an agent, then by Observation 3.3 it holds that

$$\sum_j M^{-1}_{u_1 j} \leq c_{u_1} - |\chi_{u_1}| \leq c_{u_1} - \epsilon.$$

Thus we can increase the allocation of $M^{-1}_{u_1 u_2}$ by ϵ, while not exceeding the capacity of agent u_1 (c_{u_1}). If u_1 is a good, agent u_2 can release ϵ units of good u_1 without violating any capacity constraints. For vertex u_t, we can follow a similar argument and use Observation 3.3 to show that the capacity constraint of u_t is not violated either. □

The remainder of the proof requires several preparations that are cast in the following lemmata.

Lemma 3.6. It is without loss of generality to assume that M^{-1} is such that

1. There are no cycles of zero value in G_f.
2. There is no path $P = u_1, u_2, \ldots, u_t$ of zero value such that $u_1 \neq 1$ is a source and u_t is a target.

Proof. Assume that there is a cycle or a path T in the flow decomposition of G_f such that $v(T) = 0$. Let ϵ be the smallest flow along an arc e of T. Let $\widehat{M}^{-1} = \widehat{M}^{-1}(T)$ as in Lemma 3.5, it follows from the Lemma that $\widehat{M}^{-1}$ is a valid assignment.

Furthermore, $v(\widehat{M}^{-1}) = v(M^{-1}) - xv(T) = v(M^{-1})$ and if we replace M^{-1} by $\widehat{M}^{-1}$ then the new G_f (for M and $\widehat{M}^{-1}$) is derived from the old G_f (for M and M^{-1}) by decreasing the flow along every arc of T by ϵ, and removing arcs whose flow is zero. In particular, at least one arc will be removed and no new arcs added. We repeat this process until G_f does not contain any cycle or path of zero value, as required. □

Thus, in the sequel we assume that M^{-1} satisfies conditions (1) and (2) of Lemma 3.6[2].

Lemma 3.7. The graph G_f does not contain a cycle.

Proof. Assume that G_f contains a cycle C which carries $\epsilon > 0$ flow.

If $v(C) < 0$, let $\widehat{M} = \widehat{M}(C)$. According to Lemma 3.5, $\widehat{M}$ is a valid assignment. The value of $\widehat{M}$ is

$$v(\widehat{M}) = v(M) - \epsilon v(C) > v(M) \ ,$$

which contradicts the maximality of M.

If $v(C) > 0$, let $\widehat{M}^{-1} = \widehat{M}^{-1}(C)$. According to Lemma 3.5, $\widehat{M}^{-1}$ is a valid assignment. We now show that $\widehat{M}^{-1}$ allocates nothing to agent 1. For there to be an edge $j \to 1$ in G_f, it must be that $M_{1j} - M^{-1}_{1j} < 0$, but $M_{1j} \geq 0$ and M^{-1} does not contain agent 1 at all so $M^{-1}_{1j} = 0$. It follows

[2]Since Inequality (8) depends only on the value of M^{-1} it does not matter which M^{-1} we work with.

that agent 1 has no incoming flow in G_f, and thus cannot be part of any cycle. The value of $\widehat{M}^{-1}$ is

$$v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(C) > v(M^{-1}) \ ,$$

which contradicts the maximality of M^{-1}.

Also, by Lemma 3.6, there are no cycles of value zero in G_f and this concludes the proof. □

In particular, Lemma 3.7 implies that there are no cycles in our flow decomposition. We next show that the only source vertex in G_f is the vertex corresponding to agent 1.

Lemma 3.8. The vertex that corresponds to agent 1 is the unique source vertex.

Proof. Proof via reduco ad absurdum. Consider the graph G_f, let $u_1 \neq 1$ be a source vertex, u_t a target vertex, and, by assumption, let $P = u_1, u_2, \ldots u_t$, be some path in G_f with flow $\epsilon > 0$. Since the vertex corresponding to agent 1 has no incoming arcs, P does not contain vertex 1. According to Lemma 3.6, such a path P cannot have value zero.

According to Lemma 3.5, the allocations $\widehat{M}^{-1} = \widehat{M}^{-1}(P)$ and $\widehat{M}(P)$ are valid (preserve capacity constraints).

Consider the following two cases.

case a: $v(P) > 0$. It follows that

$$v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(P) > v(M^{-1}).$$

case b: $v(P) < 0$. It follow that

$$v(\widehat{M}) = v(M) - \epsilon v(P) > v(M).$$

In both cases we've reached contradiction, either to the optimality of M^{-1} (case a) or to the optimality of M (case b). □

Lemma 3.8 implies that all the paths in our flow decomposition originate at agent 1. We are now ready to describe the construction of the allocation D^{-2}:

1. Stage I: initially, $D^{-2} := M^{-1}$.
2. Stage II: for every good j, let $x = \min\{M_{2j}, M^{-1}_{2j}\}$, and set $D^{-2}_{2j} := M^{-1}_{2j} - x$ and $D^{-2}_{1j} := x$.
3. Stage III: for every flow path P in the flow decomposition of G_f that contains agent 2, let $\hat{P}$ be the prefix of P up to agent 2. For every agent to good arc $(i \to j) \in \hat{P}$ set $D^{-2}_{ij} := D^{-2}_{ij} + f(P)$, and for every good to agent arc $(j \to i) \in \hat{P}$ set $D^{-2}_{ij} := D^{-2}_{ij} - f(P)$.

It is easy to verify that D^{-2} indeed does not allocate any good to agent 2. Also, the allocation to agent 1 in D^{-2} is of the same size as the allocation to agent 2 in M^{-1}. Since $c_1 \geq c_2$, D^{-2} is a valid allocation.

To conclude the proof of Theorem 3.2 we now show that:

Lemma 3.9. Allocation D^{-2} satisfies (8).

Proof. Rearranging (8), we obtain

$$\begin{aligned} v(D^{-2}) &\geq v(M^{-1}) && (13)\\ &+ \sum_{j=1}^{s} (v_1(j) - v_2(j)) \cdot \min(M_{2j}, M_{2j}^{-1}) && (14)\\ &+ \sum_{j:M_{2j}>M_{2j}^{-1}} (v_1(j) - v_2(j))\,(M_{2j} - M_{2j}^{-1}). && (15) \end{aligned}$$

At the end of stage I, we have $D^{-2} = M^{-1}$ and so the inequality above at line (13) (Excluding expressions (14) and (15)) holds trivially. It is also easy to verify that at the end of stage II, the inequality above that spans expressions (13) and (14) (and excludes expression (15)) holds. What we show next is that at the end of stage III, the full inequality above will hold.

Consider a good j such that $M_{2j} > M_{2j}^{-1}$. In G_f we have an arc $2 \to j$ such that $f_{2\to j} = M_{2j} - M_{2j}^{-1}$, therefore in the flow decomposition we must have paths $P_1, \ldots, P_\ell$, all containing the arc $2 \to j$, such that

$$\sum_{k=1}^{\ell} f(P_k) = f_{2\to j} = M_{2j} - M_{2j}^{-1}. \tag{16}$$

For every $k = 1, \ldots, \ell$, let $\widehat{P}_k$ denote the prefix of P_k up to agent 2. Consider the cycle C consisting of $\widehat{P}_k$ followed by arcs $2 \to j$ and $j \to 1$. We claim that the value of this cycle is non-negative.

Consider the allocation $\widehat{M}(C)$ which is a valid allocation from Lemma 3.5. Observe that $v(\widehat{M}) = v(M) - \epsilon v(C) > v(M)$. This now contradicts the assumption that M maximizes v over all allocations. We obtain

$$v(\widehat{P}_k) + v_2(j) - v_1(j) \geq 0.$$

Rearranging and multiplying by $f(P_k)$, it follows that

$$f(P_k)v(\widehat{P}_k) \geq f(P_k)\left(v_1(j) - v_2(j)\right).$$

Summing over all paths $k = 1, \ldots, \ell$, we get

$$\sum_{k=1}^{\ell} \Big(f(P_k)v(\widehat{P}_k)\Big) \geq (v_1(j) - v_2(j)) \sum_{k=1}^{\ell} f(P_k).$$

Substituting (16) in the last inequality establishes the following inequality:

$$\sum_{k=1}^{\ell} \big(f(P_k)v(\widehat{P}_k)\big) \geq \big(v_1(j) - v_2(j)\big)\big(M_{2j} - M_{2j}^{-1}\big). \tag{17}$$

The left hand side of (17) is exactly the gain in value of the allocation when applying stage III to the paths $\widehat{P}_1, \ldots, \widehat{P}_\ell$ during the construction of D^{-2} above. The right hand side is the term which we add in (15).

To conclude the proof of Lemma 3.9, we note that stage III may also deal with other paths that start at agent 1 and terminate at agent 2. Such paths must have non-negative value and

thus can only increase the value of D^{-2}. Otherwise we can construct an allocation $\tilde{M}$, such that $v(\tilde{M}) > v(M)$ by decreasing M_{ij} by ϵ for every arc $(i \to j) \in P$ and increasing M_{ij} by ϵ for every arc $(j \to i) \in P$ as we did constructing $\widehat{M}$ in Lemma 3.5. The allocation $\tilde{M}$ is valid since it preserves capacities of vertices that are internal on the path and decreases only arcs with flow on them, $M_{ij} > \epsilon$. Finally, the capacity of a source agent vertex can be increased according to Observation 3.3. □

This concludes the proof of Theorem 3.2. □

The following is a direct corollary of Theorem 3.2.

Corollary 3.10. If all agent capacities are equal, then the VCG allocation with Clarke-pivot payments is EF.

4 Heterogeneous Capacities: IC + EF imply Positive Transfers

Do Clarke-pivot payments work also under heterogeneous capacities? The answer is no, as demonstrated in Example 1.1. In this section we prove a stronger result, showing that any mechanism that is both incentive compatible and envy-free for heterogeneous capacities must have positive transfers. We remark that IC, NPT, and IR $\Leftrightarrow$ Clarke-pivot payments, which, along with Example 1.1 implies that one cannot have an efficient mechanism that is IC, NPT, IR, and EF for heterogeneous capacities, here we prove that even without the individual rationality requirement, this is impossible.

Theorem 4.1. Consider capacitated valuations with heterogeneous capacities such that the number of goods exceeds the smallest agent capacity. There is no mechanism that is simultaneously efficient, IC, EF, and has no positive transfers. That is, any IC and EF efficient mechanism has some valuations v for which the mechanism pays an agent.

Remark: Note that the conditions on the capacities of the agents and the number of goods are necessary. If capacities are homogeneous or the total supply of goods is at most the minimum agent capacity, then Clarke-pivot payments, that are known to be incentive compatible, individually rational, and have no positive transfers, are also envy-free.

Proof. We start with a warm-up of capacitated valuations with two agents and two goods where agent $i = 1, 2$ has capacity i. We then generalize the proof to arbitrary heterogeneous settings. To ease the notation we abbreviate $v_i(j)$ to v_{ij}.

Two agents and two goods:

One can easily verify that the social optimum is as follows. (we omit cases with ties).[3]

- If $v_{21} > v_{11}$ and $v_{22} > v_{12}$, then $\mathrm{Opt}_2 = \{1, 2\}$ and $\mathrm{Opt}_1 = \emptyset$. We refer to this class of valuations as class A.

[3] The social optimum is unique when there are no ties. Valuations v's with ties form a lower dimensional measure 0 set. It suffices to consider valuations without ties for both existence or non-existence claims of IC or EF payments. This is clear for non-existence, for existence, the payments for a v with ties is defined as the limit when we approach this point through v's without ties that result in the same allocation. Clearly IC and EF properties carry over, also IR and NPT.

- If $v_{11} - v_{21} > \max\{0, v_{12} - v_{22}\}$, then $\text{Opt}_1 = \{1\}$ and $\text{Opt}_2 = \{2\}$. We refer to this class of valuations as class B_1.
- If $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$, then $\text{Opt}_1 = \{2\}$ and $\text{Opt}_2 = \{1\}$. We refer to this class of valuations as class B_2.

$x+3\epsilon$	$x+\epsilon$
0	0

(a)

$x+3\epsilon$	$x+\epsilon$
$x+\epsilon$	x

(b)

0	0
$x+\epsilon$	x

(c)

Figure 3: These matrices correspond to three valuation profiles, where in each matrix the rows correspond to the agents and the columns correspond to goods. The valuations in matrices (a) and (b) belong to class B_1, and the valuation in matrix (c) belongs to class A.

Substituting the above in Proposition 2.2 we obtain that for $v \in B_1$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \leq v_2(\text{Opt}_2) - v_1(\text{Opt}_2) = v_{22} - v_{12} \ , \tag{18}$$

and agent 2 does not envy agent 1 if and only if

$$h_2(v_1) - h_1(v_2) \leq v_1(\text{Opt}_1) - v_2(\text{Opt}_1) = v_{11} - v_{21} \ . \tag{19}$$

Fix an $\epsilon > 0$, and some $x > 0$, and consider the valuation v where $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$ and $v_{21} = v_{22} = 0$ (see Figure 4(a)). This valuation is clearly in B_1. Substituting in (19), agent 2 does not envy agent 1 in v if and only if

$$-(x + 3\epsilon) \leq h_1(0,0) - h_2(x + 3\epsilon, x + \epsilon). \tag{20}$$

Next consider the valuation v where $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = x + \epsilon$, and $v_{22} = x$ (see Figure 4(b)). This valuation is clearly in B_1 as well. Substituting in (18), agent 1 does not envy agent 2 in v if and only if

$$h_1(x + \epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \leq x - (x + \epsilon) = -\epsilon. \tag{21}$$

Combining (20) and (21), it follows that

$$h_1(x + \epsilon, x) \leq h_2(x + 3\epsilon, x + \epsilon) - \epsilon \leq h_1(0,0) + x + 2\epsilon \tag{22}$$

The no positive transfers requirement states that for any v, $p_i \geq 0$ for every i; in particular, $p_1 = h_1(v_2) - v_2(\text{Opt}_2) \geq 0$; i.e.,

$$h_1(v_2) \geq v_2(\text{Opt}_2) \ . \tag{23}$$

Finally, consider the valuations v where $v_{11} = v_{12} = 0$, $v_{21} = x+\epsilon$, and $v_{22} = x$ (see Figure 4(c)). Clearly, the optimal allocation is one in which agent 2 gets both goods, thus $v \in A$ and $v_2(\text{Opt}_2) = 2x + \epsilon$. From (22) it follows that $h_1(x + \epsilon, x) \leq h_1(0,0) + x + 2\epsilon$. From (23) it follows that $h_1(x + \epsilon, x) \geq v_2(\text{Opt}_2) = 2x + \epsilon$. Combining we obtain $h_1(0,0) \geq x - \epsilon$. However, $h_1(0,0)$ cannot be a function of x; in particular if $h_1(0,0) < x - \epsilon$, then we obtain a contradiction.

This simple case gives us essentially all the intuition and structure that is required for solving the general case.

Heterogeneous capacities, multiple agents and goods:

Let c be the smallest agent capacity and rename the agents such that c is the capacity of agent 1, and the capacity of agent 2 is strictly greater than c. Consider an instance with at least $c+1$ goods, and valuation functions satisfying $v_{ij} = 0$ if $i > 2$ or $j > c+1$, and $v_{ij} = v_{ij'}$ for $i = 1, 2$ and every $2 \leq j, j' \leq c+1$.

It is easy to verify that an optimal allocation Opt is obtained as follows (where we omit cases with ties and only define the allocation of goods $j = 1, \ldots, c+1$).

- If $v_{21} > v_{11}$ and $v_{22} > v_{12}$, then $\mathrm{Opt}_2 = \{1, \ldots, c+1\}$ and $\mathrm{Opt}_1 = \emptyset$. We refer to this class of valuations as class A.
- If $v_{11} > v_{21}$ and $v_{12} < v_{22}$, then $\mathrm{Opt}_1 = \{1\}$ and $\mathrm{Opt}_2 = \{2, \ldots, c+1\}$. We refer to this class of valuations as class B_1.
- If $v_{11} - v_{21} > v_{12} - v_{22}$ and $v_{12} > v_{22}$, then $\mathrm{Opt}_1 = \{1, \ldots, c\}$ and $\mathrm{Opt}_2 = \{c+1\}$. We refer to this class of valuations as class B_1^+.
- If $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$, then $\mathrm{Opt}_2 = \{1\}$ and $\mathrm{Opt}_1 = \{2, \ldots, c+1\}$. We refer to this class of valuations as class B_2.

Substituting the above in proposition 2.2 we obtain that for $v \in B_1^+$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \leq v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2) = v_{22} - v_{12} \ , \tag{24}$$

and agent 2 does not envy agent 1 if and only if

$$\begin{aligned} h_2(v_1) - h_1(v_2) &\leq v_1(\mathrm{Opt}_1) - v_2(\mathrm{Opt}_1) \\ &= v_{11} + (c-1)v_{12} - v_{21} - (c-1)v_{22} \ . \end{aligned} \tag{25}$$

Fix an $\epsilon > 0$ and some $x > 0$, and consider the valuation v where $v_{11} = x + 3\epsilon$, $v_{1j} = x + \epsilon$ for $j = 2, \ldots, c+1$, and $v_{2j} = 0$ for $j = 2, \ldots, c+1$. This valuation is clearly in B_1^+. Substituting the corresponding values in (25) we obtain that agent 2 does not envy agent 1 if and only if

$$-cx - (c+2)\epsilon \leq h_1(0,0) - h_2(x + 3\epsilon, x + \epsilon) \ . \tag{26}$$

Next consider the valuation v where $v_{11} = x+3\epsilon$, $v_{1j} = x+\epsilon$ for $j = 2, \ldots, c+1$, $v_{21} = x+\epsilon$, and $v_{2j} = x$ for $j = 2, \ldots, c+1$. This valuation is clearly in B_1^+ as well. Substituting the corresponding values in (24) we obtain that agent 1 does not envy agent 2 if and only if

$$h_1(x+\epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \leq -\epsilon \ . \tag{27}$$

Combining (26) and (27) we obtain,

$$\begin{aligned} h_1(x+\epsilon, x) &\leq h_2(x + 3\epsilon, x + \epsilon) - \epsilon \\ &\leq h_1(0,0) + cx + (c+2)\epsilon - \epsilon \\ &= h_1(0,0) + cx + (c+1)\epsilon. \end{aligned} \tag{28}$$

Finally, consider the valuations v where $v_{11} = v_{1j} = 0$ for $j = 2, \ldots, c+1$, $v_{21} = x + \epsilon$, and $v_{2j} = x$ for $j = 2, \ldots, c+1$. Clearly, the optimal allocation is one in which agent 2 gets all $c+1$ goods, thus $v \in A$ and $v_2(\mathrm{Opt}_2) = (c+1)x + \epsilon$. From (28) it follows that $h_1(x+\epsilon, x) \leq h_1(0,0) + cx + (c+1)\epsilon$. In order to satisfy no positive transfers, according to (23), it holds that $h_1(x+\epsilon, x) \geq v_2(\mathrm{Opt}_2) = (c+1)x + \epsilon$. Combining we obtain $h_1(0,0) \geq x - c\epsilon$. However, $h_1(0,0)$ cannot be a function of x and c; in particular if $h_1(0,0) < x - c\epsilon$, then we obtain a contradiction. $\square$

5 IC+EF mechanism for capacitated valuations with two agents

In the previous section we showed that it is impossible to satisfy EF, IC and NPT simultaneously. Here we show that if we forego the NPT requirement, then capacitated valuations with two agents (and arbitrary capacities and number of goods) admits a mechanism which satisfies the other two properties as well as IR.

Proposition 5.1. 2-agents capacitated valuations admit a mechanism that is simultaneously IC, EF and IR.

Proof. Let c_i be the capacity of agent i and assume without loss of generality that $c_1 \leq c_2$. Given a vector $(x_1, x_2 \ldots)$ let $top_b\{x\}$ be the set of the b largest entries in x.

We show that

$$h_1(v_2) = \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \quad \text{and} \quad h_2(v_1) = \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \tag{29}$$

give VCG payments which are envy-free.

By Proposition 2.2, it is sufficient to show that for $i = 1, j = 2$ and for $i = 2, j = 1$ it holds that $h_i(v_j) - h_j(v_i) \leq v_j(\mathrm{Opt}_j) - v_i(\mathrm{Opt}_j)$. By substituting h_1 and h_2 from (29), this is equivalent to

$$\sum_{j \in top_{c_1}\{v_2\}} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \leq v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2) \tag{30}$$

and

$$\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \leq v_1(\mathrm{Opt}_1) - v_2(\mathrm{Opt}_1). \tag{31}$$

Assume first that the number of goods is exactly $c_1 + c_2$. Clearly, in the optimal solution, agent 1 will get the c_1 goods that maximize $v_{1j} - v_{2j}$ and agent 2 will get the c_2 goods that minimize this difference.

We first establish (31):

$$\begin{aligned}
\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} &\leq \sum_{j \in top_{c_1}\{v_1\}} (v_{1j} - v_{2j}) \\
&\leq \sum_{j \in top_{c_1}\{v_1 - v_2\}} (v_{1j} - v_{2j}) \\
&= \sum_{j \in \mathrm{Opt}_1} (v_{1j} - v_{2j}) \\
&= v_1(\mathrm{Opt}_1) - v_2(\mathrm{Opt}_1)
\end{aligned}$$

where the inequalities follow by the fact that for every $v_i \in \mathbb{R}^s_{\geq 0}$ and $S \subset [s]$, it holds that

$$\sum_{j \in top_{|S|}\{v_i\}} v_{ij} \geq \sum_{j \in S} v_{ij}. \tag{32}$$

In what follows we establish (30). We use the following additional notation: for a subset Y of entries, let $top_b\{x|Y\}$ denote the set of b largest entries in x projected on Y. In addition, let $S = top_{c_1}\{v_2\} \cap (\mathrm{Opt}_2 \setminus top_{c_1}\{v_1|\mathrm{Opt}_2\})$; i.e., S is the (possibly empty) set of goods that are among the top c_1 goods for v_2, are also in Opt_2, but are not among the top c_1 goods for v_1 in Opt_2.

$$\begin{aligned} v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2) &= \sum_{j \in \mathrm{Opt}_2} v_{2j} - \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} v_{1j} \\ &= \sum_{j \in \mathrm{Opt}_2 \setminus top_{c_1}\{v_1|\mathrm{Opt}_2\}} v_{2j} + \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} (v_{2j} - v_{1j}) \\ &\geq \sum_{j \in S} v_{2j} + \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} (v_{2j} - v_{1j}) \\ &\equiv \sum_{j \in S} v_{2j} + \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} (v_{2j} - v_{1j}). \end{aligned} \tag{33}$$

Let S' be a set of $|S|$ goods from $\mathrm{Opt}_1 \cup top_{c_1}\{v_1|\mathrm{Opt}_2\}$ which are not contained in $top_{c_1}\{v_2\}$. Such a set always exists because there are $2c_1$ goods in $\mathrm{Opt}_1 \cup top_{c_1}\{v_1|\mathrm{Opt}_2\}$, and exactly $c_1 - |S|$ of them are in $top_{c_1}\{v_2\}$, and therefore we have $c_1 + |S|$ goods to choose S' from. Therefore, in order to establish (30), it suffices to show that

$$\sum_{j \in S} v_{2j} + \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} (v_{2j} - v_{1j}) \geq \sum_{j \in top_{c_1}\{v_2\}} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j}.$$

This is established in what follows.

$$\begin{aligned} \sum_{j \in top_{c_1}\{v_2\}} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} &= \sum_{j \in S} v_{2j} + \sum_{j \in top_{c_1}\{v_2\} \setminus S} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \\ &\leq \sum_{j \in S} v_{2j} + \sum_{j \in S' \cup top_{c_1}\{v_2\} \setminus S} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \\ &\leq \sum_{j \in S} v_{2j} + \sum_{j \in S' \cup top_{c_1}\{v_2\} \setminus S} (v_{2j} - v_{1j}) && (34) \\ &\leq \sum_{j \in S} v_{2j} + \sum_{j \in top_{c_1}\{v_1|\mathrm{Opt}_2\}} (v_{2j} - v_{1j}) && (35) \end{aligned}$$

Inequality (34) follows from Equation (32) since $S' \cup top_{c_1}\{v_2\} \setminus S$ by definition contains exactly c_1 goods. Finally, to establish Inequality (35), observe that all the goods in $top_{c_1}\{v_1|\mathrm{Opt}_2\}$ belong to Opt_2, while all the goods that belong to $S' \cup top_{c_1}\{v_2\} \setminus S$ but not to $top_{c_1}\{v_1|\mathrm{Opt}_2\}$ belong to Opt_1. Recalling that $v_{2j} - v_{1j} \geq v_{2j'} - v_{1j'}$ for every $j \in \mathrm{Opt}_2, j' \in \mathrm{Opt}_1$ concludes the derivation of the inequality.

It remains to analyze the cases where the number of goods is different than $c_1 + c_2$. If the number of goods is less than $c_1 + c_2$, then consider a set $\mathfrak{D}$ of "dummy" goods that are added to

the set of "real" goods, such that $v_{1j} = v_{2j} = 0$ for every $j \in \mathfrak{D}$. These dummy goods do not change the optimal allocation projected on the real goods, and for every agent and every bundle, the valuation of the agent to the bundle is equal to her valuation for the set of real goods in the bundle. In addition, the values of h_1 and h_2 are also equal to their values as defined with respect to the real goods alone. Therefore, the aforementioned argument (for the case of $c_1 + c_2$ goods) can be applied here as well.

We next consider the case in which there are more than $c_1 + c_2$ goods. Observe that all the goods involved in Equations (30) and (31) participate in the optimal solution (as $top_{c_2}\{v_2\}$ and $top_{c_1}\{v_1\}$ must both be included in the optimal solution). Therefore, it is sufficient to consider the set of $c_1 + c_2$ goods that participate in the optimal solution. $\square$

6 Subadditive Valuations: IC+EF mechanism for two agents and two goods

In previous sections we restricted attention to capacitated valuations. Here, we turn to the more general family of subadditive valuations, but restrict attention to the case of two agents and two goods. For this case, we construct a mechanism that is simultaneously IC, EF and IR. This is summarized in the following proposition.

Proposition 6.1. For any subadditive allocation setting with two agents and two goods, a VCG mechanism with the following h_1, h_2 functions is envy free and individually rational:

$$\begin{aligned} h_1(v_2) &= \max(v_2(\{1\}), v_2(\{2\})) \\ h_2(v_1) &= \max(v_1(\{1\}), v_1(\{2\})) \end{aligned}$$

Proof. By Proposition 2.2, a VCG mechanism is envy free if and only if

$$h_1(v_2) - h_2(v_1) \leq v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2), \text{ and} \tag{36}$$

$$h_2(v_1) - h_1(v_2) \leq v_1(\mathrm{Opt}_1) - v_2(\mathrm{Opt}_1). \tag{37}$$

The only possible allocations are both goods allocated to same agent or each agent gets one good. Wlog., we can assume that good 2 is allocated to agent 2.

Case 1: $\mathrm{Opt}_2 = \{1, 2\}$, $\mathrm{Opt}_1 = \emptyset$.
Case 2: $\mathrm{Opt}_2 = \{2\}$, $\mathrm{Opt}_1 = \{1\}$.

We establish via case analysis that (36) and (37) hold in these two cases. To simplify presentation we use $v_i(1), v_i(2), v_i(1, 2)$ when we refer to $v_i(\{1\}), v_i(\{2\})$, and $v_i(\{1, 2\})$ respectively. In addition, we use $\max v_i$ and $\min v_i$ to denote $\max\{v_i(1), v_i(2)\}$ and $\min\{v_i(1), v_i(2)\}$ for every agent i.

Establishing (36) for Case 1: From subadditivity,

$v_1(1, 2) \leq v_1(1) + v_1(2) = \max v_1 + \min v_1$.

From optimality,

$v_2(1, 2) \geq \max\{v_1(1) + v_2(2), v_1(2) + v_2(1)\} \geq \max v_2 + \min v_1$.

Combining,

$$\begin{aligned} v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2) &= v_2(1,2) - v_1(1,2) \\ &\geq \max v_2 + \min v_1 - \max v_1 - \min v_1 \\ &= \max v_2 - \max v_1 = h_1(v_2) - h_2(v_1) \end{aligned}$$

Establishing (37) for Case 1:
From subadditivity,
$v_2(1,2) \leq v_2(1) + v_2(2) = \max v_2 + \min v_2$.
From optimality,
$v_2(1,2) \geq \max\{v_1(1) + v_2(2), v_1(2) + v_2(1)\} \geq \max v_1 + \min v_2$.
Combining together, we get $\max v_2 \geq \max v_1$ and therefore,

$$h_2(v_1) - h_1(v_2) = \max v_1 - \max v_2 \leq 0 = v_1(\mathrm{Opt}_1) - v_2(\mathrm{Opt}_1).$$

Establishing (36) for Case 2: We need to show, that

$$v_2(\mathrm{Opt}_2) - v_1(\mathrm{Opt}_2) - (h_1(v_2) - h_2(v_1)) = v_2(2) - v_1(2) - \max v_2 + \max v_1 \geq 0.$$

If $\max v_2 = v_2(2)$, then the above inequality trivially holds. If $\max v_2 = v_2(1)$, then the above inequality follows from optimality of allocation, $v_2(2) + v_1(1) \geq v_1(2) + v_2(1)$.

We omit the proof of (37) for Case 2 since it is similar to (36) for Case 2.

This establishes the assertion of the proposition. □

Recall that the valuation of an agent i for bundle B is defined as $v_i(B) \equiv \sum_{j \in top_{c_i}\{v_i|B\}} v_{ij}$, which is a special case of subadditive valuations. The following is, therefore, a direct corollary of Proposition 6.1.

Corollary 6.2. For capacitated valuations (public or private) with 2-agent and 2-goods, the VCG mechanism with the following h_1, h_2 functions is EF and IC:

$$h_1(v_2, c_2) = \begin{cases} \max(v_{21}, v_{22}) & c_2 \in \{1,2\} \\ 0 & c_2 = 0 \end{cases}$$

$$h_2(v_1, c_1) = \begin{cases} \max(v_{11}, v_{12}) & c_1 \in \{1,2\} \\ 0 & c_1 = 0 \end{cases}$$

Remark: Note that with two goods, all $c_i \geq 2$ are equivalent, therefore it is sufficient to consider capacities $\in \{0, 1, 2\}$.

7 Discussion and open problems

This work initiates the study of efficient, incentive compatible, and envy free mechanisms for capacitated valuations.

Our work suggests a host of problems for future research on heterogeneous capacitated valuations and generalizations thereof.

We know that, generally, there may be no mechanism that is both IC and EF even if we allow positive transfers [4].

First, is there a mechanism for games with more than two agents that is efficient, IC, and EF ? We conjecture that such mechanisms do exist and believe this is also the case for any combinatorial auction with subadditive valuations (which generalizes capacitated valuations with private or public capacities). We provided such mechanisms for capacitated valuations with two agents (public capacities) and for subadditive valuations with two agents and two goods.

Second, our work focused on efficient mechanisms; i.e., ones that maximize social welfare. A natural question is how well the optimal social welfare can be *approximated* by a mechanism that is IC, EF, and NPT.

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[4] As an example, consider a setting with two goods, a, b and three agents, where $v_1(a) = v_1(b) = v_1(\{a, b\})$, $v_2(a) = v_2(b) = v_2(\{a, b\})$, and $v_3(a) = v_3(b) = 0$, while $v_3(\{a, b\}) > 0$. One can easily verify that this setting has no mechanism that is simultaneously incentive compatible and envy free. This example is due to Noam Nisan.

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Truth and Envy in Capacitated Allocation Games

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Abstract

We study auctions with additive valuations where agents have a limit on the number of items they may receive. We refer to this setting as *capacitated allocation games.* We seek truthful and envy free mechanisms that maximize the social welfare. *I.e.*, where agents have no incentive to lie and no agent seeks to exchange outcomes with another.

In 1983, Leonard showed that VCG with Clarke Pivot payments (which is known to be truthful, individually rational, and have no positive transfers), is also an envy free mechanism for the special case of n items and n unit capacity agents. We elaborate upon this problem and show that VCG with Clarke Pivot payments is envy free if agent capacities are all equal. When agent capacities are not identical, we show that there is no truthful and envy free mechanism that maximizes social welfare if one disallows positive transfers.

For the case of two agents (and arbitrary capacities) we show a VCG mechanism that is truthful, envy free, and individually rational, but has positive transfers. We conclude with a host of open problems that arise from our work.

1 Introduction

We consider *allocation problems* where a set of objects is to be allocated amongst m agents, where every agent has an additive and non negative valuation function. We study mechanisms that are truthful, envy free, and maximize the social welfare (sum of valuations). The utility of an agent i is the valuation of the bundle assigned to i, $v_i(\mathrm{OPT})$, minus any payment, p_i.

A mechanism is incentive compatible (or truthful) if it is a dominant strategy for every agent to report her private information truthfully [4]. A mechanism is envy-free if no agent wishes to switch her outcome with that of another [1, 2, 9, 6, 7, 10].

Any allocation that maximizes the social welfare has payments that make it truthful — in particular — any payment of the form

$$p_i = h_i(t^{-i}) - \sum_{j \neq i} v_j(\mathrm{OPT}) \tag{1}$$

where OPT is an allocation maximizing the social welfare and t^{-i} are the types of all agents but agent i. Similarly, any allocation that maximizes the social welfare has payments that make it envy free, this follows from a characterization of envy free allocations (see [3]). Unfortunately, the set of payments that make the mechanism truthful, and the set of payments that make the mechanism envy free, need not intersect. In this

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paper we seek such payments, *i.e.*, payments that make the mechanism simultaneously truthful and envy free.

An example of a mechanism that is simultaneously truthful and envy free is the Vickrey 2nd price auction. Applying the 2nd price auction to an allocation problem assigns items successively, every item going to the agent with the highest valuation to the item at a price equal to the 2nd highest valuation. If, for example, for all items, agent i has maximal valuation, then agent i will receive all items.

Leonard [5] considered the problem of assigning people to jobs, n people to n positions, and called this problem the permutation game. The Vickrey 2nd price auction is irrelevant in this setting because no person can be assigned to more than one position. Leonard showed that VCG with Clarke Pivot payments is simultaneously truthful and envy free. Under Clarke Pivot payments, agents internalize their externalities, *i.e.*,

$$h_i(t^{-i}) = \sum_{j \neq i} v_j(\mathrm{OPT}^{-i}) \tag{2}$$

where OPT^{-i} is the optimal allocation if there was no agent i. By substituting $\sum_{j \neq i} v_j(\mathrm{OPT}^{-i})$ for $h_i(t^{-i})$ in Equation 1 one can interpret Clarke Pivot payments as though an agent pays for how much others lose by her presence, *i.e.*, the agent internalizes her externalities.

Motivated by the permutation game, we consider a more general capacitated allocation problem where agents have associated capacities. Agent i has capacity U_i and cannot be assigned more than U_i items. Like Leonard, we seek a mechanism that is simultaneously truthful and envy free. The private types we consider may include both the valuation and the capacity (private valuations and private capacities) or only the valuation (private valuations, public capacity). Leonard's proof uses LP duality and it is not obvious how to extend it to more general settings.

Before we address this question, one needs to ask what does it mean for one agent to envy another when they have different capacities? A lower capacity agent may be unable to switch allocations with a higher capacity agent. To deal with this issue, we allow agent i, with capacity less than that of agent i' to choose whatever items she desires from the i' bundle, up to her capacity. *I.e.*, we say that agent i envies agent i' if agent i prefers a subset of the allocation to agent i', along with the price set for agent i', over her own allocation and price.

The VCG mechanism (obey Equation 1) is always truthful. In fact, any truthful mechanisms that choose the socially optimal allocation in capacitated allocation problems must be VCG [8]. We obtain the following:

1. For agents with private valuations and either private or public capacities, under the VCG mechanism with Clarke Pivot payments, a higher capacity agent will never envy a lower capacity agent. In particular, if all capacities are equal then the mechanism is envy free. (See Section 3).

2. For agents with private valuations, and either private or public capacities, any envy free VCG payment must allow positive transfers. (See Section 4).

3. For two agents with private valuations and arbitrary public capacities, there exist VCG payments such that the mechanism is envy free. It follows that such payments must allow positive transfers. (See Section 5).

4. For two agents with private valuations and private capacities, and for two items, there exist VCG payments such that the mechanism is envy free. (See Section 6).

2 Preliminaries

Let U be a set of objects, and let v_i be a valuation function associated with agent i, $1 \leq i \leq m$, that maps sets of objects into $\Re$. We denote by v a sequence $< v_1, v_2, \ldots, v_m >$ of valuation functions one for each agent.

An allocation function[1] a maps a sequence of valuation functions $v =< v_1, v_2, \ldots, v_m >$ into a partition of U consisting of m parts, one for each agent. *I.e.*,

$$a(v) =< a_1(v), a_2(v), \ldots, a_m(v) >,$$

where $\cup_i a_i(v) \subseteq U$ and $a_i(v) \cap a_j(v) = \emptyset$ for $i \neq j$. A payment function[2] is a mapping from v to $\Re^m$, $p(v) =< p_1(v), p_2(v), \ldots, p_m(v) >$, $p_i(v) \in \Re$. We assume that payments are from the agent to the mechanism (if the payment is negative then this means that the transfer is from the mechanism to the agent).

A mechanism is a pair of functions, $M = \langle a, p \rangle$, where a is an allocation function, and p is a payment function. For a sequence of valuation functions $v = \langle v_1, v_2, \ldots, v_m \rangle$, the utility to agent i is defined as $v_i(a_i(v)) - p_i(v)$. Such a utility function is known as quasi-linear.

Let $v =< v_1, v_2, \ldots, v_m >$ be a sequence of valuations, we define (v_i', v^{-i}) to be the sequence of valuation functions arrived by substituting v_i by v_i', i.e.,

$$(v_i', v^{-i}) =< v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_m > .$$

We next define mechanisms that are incentive compatible, envy-free, and both incentive compatible and envy-free.

- A mechanism is *incentive compatible* (IC) if it is a dominant strategy for every agent to reveal her true valuation function to the mechanism. *I.e.*, if for all i, v, and v_i':

$$\begin{aligned} & v_i(a_i(v)) - p_i(v) \geq v_i(a_i(v_i', v^{-i})) - p_i(v_i', v^{-i}); \\ & \Leftrightarrow p_i(v) \leq p_i(v', v^{-i}) + \Big(v_i(a_i(v)) - v_i(a_i(v_i', v^{-i}))\Big). \end{aligned} \tag{3}$$

- A mechanism is *envy-free* (EF) if no agent seeks to switch her allocation and payment with another. *I.e.*, if for all $1 \leq i, j \leq m$ and all v:

$$\begin{aligned} & v_i(a_i(v)) - p_i(v) \geq v_i(a_j(v)) - p_j(v); \\ & \Leftrightarrow p_i(v) \leq p_j(v) + \Big(v_i(a_i(v)) - v_i(a_j(v))\Big). \end{aligned} \tag{4}$$

- A mechanism (a, p) is *incentive compatible and envy-free* ($IC \cap EF$) if (a, p) is both incentive compatible and envy-free.

Vickrey-Clarke-Groves (VCG) mechanism: A mechanism $M = \langle a, p \rangle$ is called a VCG mechanism if:

- $a(v) \in \text{argmax}_{a \in A} \sum_{i=1}^{m} v_i(a_i(v))$, and

[1] Here we deal with indivisible allocations, although our results also extend to divisible allocations with appropriate modifications.

[2] In this paper we consider only deterministic mechanisms and can therefore omit the allocation as an argument to the payment function.

- $p_i(v) = h_i(v^{-i}) - \sum_{j \neq i} v_j(a_j(v))$, where h_i does not depend on v_i, $i = 1, \ldots, m$.

It is known that any mechanism whose allocation function a maximizes $\sum_{i=1}^{m} v_i(a_i(v))$ (social welfare) is incentive compatible if and only if it is a VCG mechanism (See, *e.g.*, [8], Theorem 9.37). In the following we will denote by opt an allocation a which maximizes $\sum_{i=1}^{m} v_i(a_i(v))$.

The *Clarke-pivot payment* for a VCG mechanism is defined by

$$h_i(v^{-i}) = \max_{a' \in A} \sum_{j \neq i} v_j(a').$$

3 VCG with Clarke-pivot payments

A *capacitated allocation game* has m agents and n items that need to be assigned to the agents. Agent i is associated with a capacity $U_i \geq 0$, denoting the limit on the number of items she can be assigned, and each item j is associated with a capacity $Q_j \geq 0$, denoting the number of available copies of item j. The valuation $v_i(j)$ denotes how much agent i values item j, and $\sum_{j \in S} v_i(j)$ is the valuation of agent i to the bundle S.

A capacitated allocation game has a corresponding bipartite graph G, where every agent $1 \leq i \leq m$ has a vertex i associated with it on the left side, and every item $1 \leq j \leq n$ has a vertex j associated with it on the right side. The weight of the edge (i, j) is $v_i(j)$. An assignment is a subgraph of G that satisfies the capacity constraints, i.e. agent i is assigned at most U_i items and item j is assigned to at most Q_j agents. Recall that we denote by opt an assignment of maximum value. We describe opt by a matrix M where M_{ij} is the number of copies of item j allocated to agent i in opt.

For player i, the graph G^{-i} is constructed by removing the vertex associated with agent i and its incident edges from G. The assignment with maximum value in G^{-i} is defined by a matrix M^{-i}.

Let M be an assignment (either in G or in G^{-i} for some i.). We denote by M_i r the i'th row of M, $(M_{i1}, M_{i2}, \ldots, M_{in})$ which gives the bundle that agent i gets. We define $v_k(M_i) = \sum_{j=1}^{n} M_{ij} v_k(j)$ and $v(M) = \sum_{i=1}^{m} v_i(M_i)$.

The Clarke-pivot payment of agent k is

$$p_k = v(M^{-k}) - v(M) + v_k(M_k) \; . \tag{5}$$

The main result of this section is that in a VCG mechanism with Clarke-pivot payments, no agent will ever envy a lower-capacity agent. In particular, this says that if all agents have the same capacity, the VCG mechanism with Clarke-pivot payments is both incentive compatible and envy-free.

The proof of our main result (Theorem 3.1) is given in terms of a factional assignment but also holds for integral assignments.

Special case of capacitated allocation games, in which there are n items and n agents, and each agent can get at most a single item was first introduced in a paper by Leonard [5], and was called a *permutation game*. Leonard proved Theorem 3.1 for this special case only, and its proof technique does not seem to generalize for larger capacities. Our proof is different.

Here is our main theorem.

Theorem 3.1. *Consider a VCG mechanism consisting of an optimal allocation M and Clarke-pivot payments (5). Then if $U_i \geq U_j$, agent i does not envy agent j.*

Let agent 1 and agent 2 be arbitrary two agents such that the capacity of agent 1 is $\geq$ that of agent 2, that is $U_1 \geq U_2$.

Let M be an optimal assignment, M^{-1} an optimal assignment without agent 1, and M^{-2} some optimal assignment without agent 2. Agent 1 does *not* envy agent 2 iff

$$v_1(M_1) - p_1 \geq v_1(M_2) - p_2$$

Based on Equation 5, this is true when:

$$\begin{aligned}
&v_1(M_1) - (v(M^{-1}) - v(M) + v_1(M_1)) = \\
&v(M) - v(M^{-1}) \geq \\
&v_1(M_2) - (v(M^{-2}) - v(M) + v_2(M_2)) = \\
&v_1(M_2) + v(M) - v(M^{-2}) - v_2(M_2)
\end{aligned}$$

Rearranging we obtain that agent 1 does not envy agent 2 iff

$$v(M^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2). \tag{6}$$

We prove the theorem by establishing (6). We use the assignments M and M^{-1} to construct an assignment D^{-2} on G^{-2} such that

$$v(D^{-2}) \geq v(M^{-1}) + v_1(M_2) - v_2(M_2)\ . \tag{7}$$

From the optimality of M^{-2}, $v(M^{-2}) \geq v(D^{-2})$, which combined with (7) implies (6).

Given assignments M and M^{-1}, we construct a flow f on an associated bipartite digraph, G_f, with vertices for every agent and item. We define arcs and flows on arcs in G_f for every agent i and item j:

- If $M_{ij} - M^{-1}_{ij} > 0$ then G_f includes an arc $i \to j$ with flow $f_{i \to j} = M_{ij} - M^{-1}_{ij}$.
- If $M_{ij} - M^{-1}_{ij} < 0$ then G_f includes an arc $j \to i$ with flow $f_{j \to i} = M^{-1}_{ij} - M_{ij}$.
- If $M_{ij} = M^{-1}_{ij}$ then G_f contains neither $i \to j$ not $j \to i$.

We define the *excess* of an agent i in G_f, and the *excess* of an item j in G_f, to be

$$\begin{aligned}
ex_i &= \sum_{(i\to j)\in G_f} f_{i\to j} - \sum_{(j\to i)\in G_f} f_{j\to i} = \sum_j \left(M_{ij} - M^{-1}_{ij}\right), \\
ex_j &= \sum_{(j\to i)\in G_f} f_{j\to i} - \sum_{(i\to j)\in G_f} f_{i\to j} = \sum_i \left(M^{-1}_{ij} - M_{ij}\right),
\end{aligned}$$

respectively.

In other words the excess is the difference between the amount flowing out of the vertex and the amount flowing into the vertex. Clearly the sum of all excesses is zero. We say that a node is a *source* if its excess is positive and we say that a node is a *target* if its excess is negative.

Observation 3.2. To summarize,

$$
\begin{aligned}
i \text{ is an agent and a source} \quad &\Rightarrow \\
0 \le \sum_j M^{-1}_{ij} + |ex_i| \quad &= \quad \sum_j M_{ij} \le U_i; \qquad (8)\\
i \text{ is an agent and a target} \quad &\Rightarrow \\
0 \le \sum_j M_{ij} + |ex_i| \quad &= \quad \sum_j M^{-1}_{ij} \le U_i; \qquad (9)\\
j \text{ is an item and a source} \quad &\Rightarrow \\
0 \le \sum_i M_{ij} + |ex_j| \quad &= \quad \sum_i M^{-1}_{ij} \le Q_j; \qquad (10)\\
j \text{ is an item and a target} \quad &\Rightarrow \\
0 \le \sum_i M^{-1}_{ij} + |ex_j| \quad &= \quad \sum_i M_{ij} \le Q_j. \qquad (11)
\end{aligned}
$$

By the standard flow decomposition theorem we can decompose f into simple paths and cycles where each path connects a source to a target. Each path and cycle T has a positive flow value $f(T) > 0$ associated with it. Given an arc $x \to y$, if we sum the values $f(T)$ of all paths and cycles T including $x \to y$ then we obtain $f_{x \to y}$.

Notice that $M^{-1}_{1j} = 0$ for all j and therefore $f_{1 \to j} \ge 0$ for all j. It follows that there are no arcs of the form $j \to 1$ in G_f.

Observation 3.3. For each path $P = u_1, u_2, \ldots, u_t$ in flow decomposition G_f, where u_1 is a source and u_t is a target, we have $f(P) \le \min\{ex_{u_1}, |ex_{u_t}|\}$.

We define the value of a path or a cycle $T = u_1, u_2, \ldots, u_t$ in G_f, to be

$$v(P) = \sum_{\substack{\text{agent } u_i,\\ \text{item } u_{i+1}}} v_{u_i}(u_{i+1}) - \sum_{\substack{\text{item } u_i,\\ \text{agent } u_{i+1}}} v_{u_{i+1}}(u_i).$$

It is easy to verify that the $\sum_T f(T) \cdot v(T)$ over all paths and cycles in our decomposition is $v(M) - v(M^{-1})$.

Lemma 3.4. *Without loss of generality, we can assume that M^{-1} is such that*

1. *There are no cycles of zero value in G_f.*
2. *There is no path $P = u_1, u_2, \ldots, u_t$ of zero value such that $u_1 \ne 1$ is a source and u_t is a target.*

Proof. Assume that there is a cycle or a path T in the flow decomposition of G_f such that $v(T) = 0$. Let x be the smallest flow along an arc e of T. We modify M^{-1} as follows: For every agent to item arc $i \to j \in T$ we increase M^{-1}_{ij} by x and for every item to agent arc $j \to i \in T$ we decrease M^{-1}_{ij} by x. Let the resulting flow be $\tilde{M}^{-1}$.

If T is a cycle then the capacity constraints are clearly preserved. If T is not a cycle, then the capacity constraints are trivially preserved for all nodes other than u_1 and u_t. From Equation (8) we know that

$$\sum_j M^{-1}_{u_1 j} \quad \le \quad U_{u_1} - |ex_{u_1}| \le U_{u_1} - x \text{ if } u_1 \text{ is an agent.}$$

Ergo, if u_1 is an agent we can increase the allocation of $M^{-1}_{u_1 u_2}$ by x, while not exceeding the capacity of agent u_1 (U_{u_1}). If u_1 is an item, agent u_2 can release x units of item u_1 without violating any capacity constraints.

We can similarly see that the capacities constraints of u_t are not violated (Equation (11)).

Furthermore $v(\tilde{M}^{-1}) = v(M^{-1}) - xv(T) = v(M^{-1})$ and if we replace M^{-1} by $\tilde{M}^{-1}$ then G_f changes by decreasing the flow along every arc of T by x, and removing arcs whose flow becomes zero (in particular at least one arc will be removed). This process does not introduce any new edges to G_f.

We repeat the process until G_f does not contain zero cycles or paths as defined. □

From now on we assume that M^{-1} is chosen according to Lemma 3.4 [3].

Lemma 3.5. *The flow f in G_f does not contain cycles.*

Proof. Assume that f contains a cycle C which carries $\epsilon > 0$ flow. Clearly C does not contain agent 1 since there is not any arc entering agent 1 in G_f.

Assume first that $v(C) < 0$. Create an assignment $\widehat{M}$ from M by decreasing M_{ij} by ϵ for each agent to item arc $i \to j \in C$ and increasing M_{ij} by ϵ for each item to agent arc $j \to i \in C$. This can be done because $M - M^{-1}$ has a flow of ϵ along the agent to item arc $i \to j$, so, it must be that $M_{ij} \geq \epsilon$. Similarly, $M - M^{-1}$ has a flow of ϵ along item to agent arcs $j \to i$ so it must be the $M_{ij} \leq U_i - \epsilon$. Since C is a cycle the assignment $\widehat{M}$ still satisfies the capacity constraints. Furthermore $v(\widehat{M}) = v(M) - \epsilon v(C) > v(M)$ which contradicts the maximality of M.

If $v(C) > 0$ we create assignment $\widehat{M}^{-1}$ from M^{-1} as follows. For every item to agent arc $j \to i \in C$ we decrease M^{-1}_{ij} by ϵ and for every agent to item arc $i \to j \in C$ we increase M^{-1}_{ij} by ϵ. This can be done because $M^{-1} - M$ has a flow of ϵ along the item to agent arc $j \to i$, so, it must be that $M^{-1}_{ij} \geq \epsilon$. Since C is a cycle $\widehat{M}^{-1}$ still satisfies the capacity constraints. Furthermore $v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(C) > v(M^{-1})$ which contradicts the maximality of M^{-1}.

We need to argue that $\widehat{M}^{-1}$ makes no assignment to agent 1, this follows because agent 1 has no incoming flow in G_f and cannot lie on any cycle.

By assumption, there no cycles of value zero in G_f. □

In particular Lemma 3.5 implies that there are no cycles in our flow decomposition.

Lemma 3.6. *Agent 1 is the only source node.*

Proof. We give a proof by contradiction, assume some other node, $u_1 \neq 1$, is a source. Then, there is a flow path $P = u_1, u_2, \ldots u_t$ from that node to a target node u_t. Since there are no arcs incoming into vertex 1, the path P cannot include agent 1.

Let ϵ be the flow along the path P in the flow decomposition.

If $v(P) > 0$ define $\widehat{M}^{-1}_{ij} = M^{-1}_{ij} + \epsilon$ for each agent to item arc $i \to j$ in P and $\widehat{M}^{-1}_{ij} = M^{-1}_{ij} - \epsilon$ for each item to agent arc $j \to i$ in P. For all other item/agent pairs (i, j), let $\widehat{M}^{-1}_{ij} = M^{-1}_{ij}$. We have that

$$v(\widehat{M}^{-1}) = v(M^{-1}) + \epsilon v(P) > v(M^{-1})$$

this would contradict the maximality of M^{-1} if $\widehat{M}^{-1}$ is a legal assignment.

[3]Since Equation (7) depends only on the value of M^{-1} it does not matter which M^{-1} we work with

If $v(P) < 0$ define $\widehat{M}_{ij} = M_{ij} - \epsilon$ for each agent to item arc $i \to j$ in P and $\widehat{M}_{ij} = M_{ij} + \epsilon$ for each item to agent arc $j \to i$ in P. For all other item/agent pairs (i, j), let $\widehat{M}_{ij} = M_{ij}$. We have that

$$v(\widehat{M}) = v(M) - \epsilon v(P) > v(M)$$

which contradicts the maximality of M.

We still need to argue that the assignment $\widehat{M}^{-1}$ (if $v(P) > 0$) and the assignment $\widehat{M}$ (if $v(P) < 0$) are legal. Because P has a flow of ϵ, $M^{-1}_{ij} \geq \epsilon$ for each item to agent arc $j \to i$ along P, and $M_{ij} \geq \epsilon$ for each agent to item arc $i \to j$ along P.

We also worry about exceeding capacities at the endpoints of P, since the size of assignments of agents/items that are internal to the path do not change.

We increase the capacity of u_1 while constructing M^{-1} only if u_1 is an agent, and increase the capacity of u_t while constructing M^{-1} only if it is an item. By Observation 3.2 this is legal. A similar argument shows that in $\widehat{M}$ the assignment of u_1 and u_t is smaller than their capacities.

According to the way we choose M^{-1}, it cannot be that $v(P) = 0$ and that P carries a flow in G_f. □

In particular Lemma 3.6 implies that all the paths in our flow decomposition start at agent 1.

We construct D^{-2} from M^{-1} as follows.

1. Stage I: Initially, $D^{-2} := M^{-1}$.
2. Stage II: For each item j let $x = \min\{M_{2j}, M^{-1}_{2j}\}$. Set $D^{-2}_{2j} := M^{-1}_{2j} - x$ and $D^{-2}_{1j} := x$.
3. Stage III: For each flow path P in the flow decomposition of G_f that contains agent 2 we consider the prefix of the path up to agent 2. For each agent to item arc $i \to j$ in this prefix we set $D^{-2}_{ij} := D^{-2}_{ij} + f(P)$, and for each item to agent arc $j \to i$ in this prefix we set $D^{-2}_{ij} := D^{-2}_{ij} - f(P)$.

It is easy to verify that D^{-2} indeed does not assign any item to agent 2. Also, the assignment to agent 1 in D^{-2} is of the same size as the assignment to agent 2 in M^{-1}. Since $U_1 \geq U_2$, D^{-2} is a legal assignment.

Lemma 3.7. *The assignment D^{-2} satisfies Equation (7).*

Proof. Rearranging Equation (7)

$$\begin{aligned} v(D^{-2}) \geq\ & v(M^{-1}) && (12)\\ & + \sum_{j=1}^{n} (v_1(j) - v_2(j)) \cdot \min(M_{2j}, M^{-1}_{2j}) && (13)\\ & + \sum_{j \mid M_{2j} > M^{-1}_{2j}} (v_1(j) - v_2(j)) \cdot (M_{2j} - M^{-1}_{2j}). && (14) \end{aligned}$$

At the end of stage I, we have $D^{-2} = M^{-1}$ and so the inequality above at line (12) (without adding (13) and (14)) holds trivially. It is also easy to verify that at the end of stage II, the inequality above that spans (12) and (13) but without (14) holds. Finally, at the end of stage III, the full inequality in (12), (13) and (14) will hold as we explain next.

Consider an item j such that $M_{2j} > M^{-1}_{2j}$. In G_f we have an arc $2 \to j$ such that $f_{2\to j} = M_{2j} - M^{-1}_{2j}$. Therefore in the flow decomposition we must have paths $P_1, \ldots, P_\ell$ all containing $2 \to j$ such that

$$\sum_{k=1}^{\ell} f(P_k) = f_{2\to j} = M_{2j} - M^{-1}_{2j} \qquad (15)$$

Let $\widehat{P}_k$ be the prefix of P_k up to agent 2. Consider the cycle C consisting of $\widehat{P}_k$ followed by $2 \rightarrow j$ and $j \rightarrow 1$. It has to be that that value of this cycle is non-negative. (Otherwise, construct $\widehat{M}$ by decreasing each agent to item arc $i \rightarrow j$ on the cycle $\widehat{M}_{ij} = M_{ij} - \epsilon$ and increasing each item to agent arc $j \rightarrow i$ on the cycle $\widehat{M}_{ij} = M_{ij} + \epsilon$. It follows, that $v(\widehat{M}) = v(M) - \epsilon v(C) > v(M)$ in contradiction of maximality of M. The matching $v(\widehat{M})$ is legal since it preserves capacities and decreases assignment associated with arcs with flow on them.)

Therefore,

$$\begin{aligned} v(\widehat{P}_k) + v_2(j) - v_1(j) &\geq 0; \\ \Rightarrow v(\widehat{P}_k) &\geq (v_1(j) - v_2(j)); \\ \Rightarrow f(P_k)v(\widehat{P}_k) &\geq f(P_k)(v_1(j) - v_2(j)); \\ \Rightarrow \sum_{k=1}^{\ell} \big(f(P_k)v(\widehat{P}_k)\big) &\geq (v_1(j) - v_2(j)) \sum_{k=1}^{\ell} f(P_k). \end{aligned}$$

Substituting Equation (15) into the above gives us that

$$\sum_{k=1}^{\ell} \big(f(P_k)v(\widehat{P}_k)\big) \geq \big(v_1(j) - v_2(j)\big)\big(M_{2j} - M_{2j}^{-1}\big). \tag{16}$$

The left hand side of equation (16) is exactly the gain in value of the matching when applying stage III to the paths $\widehat{P}_1, \ldots, \widehat{P}_\ell$ during the construction of D^{-2} above. The right hand side is the term which we add in Equation (14).

To conclude the proof of Lemma 3.7, we note that stage III may also deal with other paths that start at agent 1 and terminate at agent 2. Such paths must have value ≥ 0 and thus can only increase the value of the matching D^{-2}. (Otherwise we can build assignment $\widehat{M}$, such that $v(\widehat{M}) > v(M)$ by decreasing M_{ij} by ϵ for each arc $i \rightarrow j \in P$ and increasing M_{ij} by ϵ for each arc $j \rightarrow i \in P$ as we did before. The matching $\widehat{M}$ is legal since it preserves capacities on inner nodes of the path, decreases only arcs with flow on them, $M_{ij} > \epsilon$. Capacity of a source agent node can be increased according to Observation 3.2.) $\square$

Corollary 3.8. *If all agent capacities are equal then the VCG allocation with Clarke-pivot payments is envy-free.*

Do Clarke-pivot payments work also under heterogeneous capacities? The answer is no. This follows since in the next section we show that any mechanism that is both incentive compatible and envy-free must have positive transfers, and Clarke-pivot payments do not.

4 Heterogeneous capacities: IC∩EF payments imply positive transfers

Consider an arbitrary VCG mechanism. Let

$$opt =< opt_1, opt_2, \ldots, opt_n >$$

denote the allocation and let

$$p_i = h_i(v^{-i}) - v^{-i}(opt) \tag{17}$$

be the payments, where

$$v^{-i}(opt) = \sum_{\substack{1 \le j \le n \\ j \neq i}} v_j(opt_j).$$

Let $v(opt) = \sum_{j=1}^{n} v_j(opt_j)$ and let

$$opt^{-i} =< opt_1^{-1}, opt_2^{-1}, \ldots, opt_{i-1}^{-i}, \emptyset, opt_{i+1}^{-i}, \ldots, opt_n >,$$

be the allocation maximizing

$$v^{-i}(opt^{-i}) = \sum_{\substack{1 \le j \le n \\ j \neq i}} v_j(opt_j^{-i}).$$

We substitute the VCG payments (17) into the envy-free conditions (4) and obtain that i does not envy j if and only if

$$\begin{aligned}
& v_i(opt_j) - p_j \le v_i(opt_i) - p_i \\
& \Leftrightarrow p_i - p_j \le v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - v^{-i}(opt) - \big(h_j(v^{-j}) - v^{-j}(opt)\big) \\
& \qquad \le v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \\
& \qquad \le v^{-i}(opt) - v^{-j}(opt) + v_i(opt_i) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \\
& \qquad \le v(opt) - (v(opt) - v_j(opt_j)) - v_i(opt_j) \\
& \Leftrightarrow h_i(v^{-i}) - h_j(v^{-j}) \le v_j(opt_j) - v_i(opt_j). \qquad (18)
\end{aligned}$$

Theorem 4.1. *Consider a capacitated allocation game with heterogeneous capacities such that the number of items exceeds the smallest agent capacity. There is no mechanism that simultaneously optimizes the social welfare, is IC∩EF, and has no positive transfers (the mechanism never pays the agents). That is, any IC∩EF mechanism has some valuations v for which the mechanism pays an agent.*

Note that the conditions on the capacities of the agents and the number of items are necessary – If capacities are homogeneous or the total supply of items is at most the minimum agent capacity then Clarke-pivot payments, that are known to be incentive compatible, individually rational, and have no positive transfers, are also envy-free.

In the rest of this section we prove Theorem 4.1. We start with a capacitated allocation game with two agents and two items where agent i has capacity i ($i = 1, 2$). We then generalize the proof to arbitrary heterogeneous games.

To ease the notation we abbreviate in the rest of the paper $v_i(j)$ to v_{ij}.

We partition the valuations into three sets A, B_1, and B_2 as follows (we omit cases with ties).[4]

[4]The optimal allocation that maximizes social welfare is uniquely defined when there are no ties. Valuations v's with ties form a lower dimensional measure 0 set. It suffices to consider valuations without ties for both existence or non-existence claims of IC or EF payments. This is clear for non-existence, for existence, the payments for a v with ties is defined as the limit when we approach this point through v's without ties that result in the same allocation. Clearly IC and EF properties carry over, also IR and nonnegativity of payments.

- (A) $v_{21} > v_{11}$ and $v_{22} > v_{12}$. For these valuations in an optimal allocation agent 2 obtains the bundle $\{1, 2\}$ and agent 1 obtains the empty bundle.
- (B_1) $v_{11} - v_{21} > \max\{0, v_{12} - v_{22}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 1 and item 2 to agent 2.
- (B_2) $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 2 and item 2 to agent 1.

Substituting the above in (18) we obtain that for $v \in B_1$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \leq v_2(opt_2) - v_1(opt_2) = v_{22} - v_{12} .$$

Agent 2 does not envy agent 1 if and only if

$$h_2(v_1) - h_1(v_2) \leq v_1(opt_1) - v_2(opt_1) = v_{11} - v_{21} .$$

Combining we obtain that there is no envy for $v \in B_1$, if and only if

$$v_{21} - v_{11} \leq h_1(v_2) - h_2(v_1) \leq v_{22} - v_{12} . \tag{19}$$

For a fixed $\epsilon > 0$, and $x > 5\epsilon$, the valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = v_{22} = 0$ is clearly in B_1. Substituting in (19) we obtain

$$-(x + 3\epsilon) \leq h_1(0, 0) - h_2(x + 3\epsilon, x + \epsilon) \leq -(x + \epsilon) \tag{20}$$

.

The valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = x + \epsilon$, and $v_{22} = x$ is also clearly in B_1 and from (19) we obtain

$$x + \epsilon - (x + 3\epsilon) \leq h_1(x + \epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \leq x - (x + \epsilon)$$

hence

$$-2\epsilon \leq h_1(x + \epsilon, x) - h_2(x + 3\epsilon, x + \epsilon) \leq -\epsilon . \tag{21}$$

Combining (20) and (21) we obtain

$$h_1(x + \epsilon, x) \leq h_2(x + 3\epsilon, x + \epsilon) - \epsilon \leq h_1(0, 0) + x + 3\epsilon \tag{22}$$

The no positive transfers requirement is that for any v,

$$h_1(v_2) \geq v_2(opt_2) . \tag{23}$$

Consider now the valuations v such that $v_{21} = x+\epsilon$, $v_{22} = x$, $v_{11} = v_{12} = x-\epsilon$. Clearly, $v \in A$ (agent 2 gets both items), hence $v_2(opt_2) = 2x-\epsilon$. Substituting this and (22) in (23) we obtain $2x-\epsilon \leq h_1(0, 0)+x+3\epsilon$, hence $h_1(0, 0) \geq x - 4\epsilon$. Clearly, for valuations with large enough x we obtain a contradiction, that is, there exist valuations where the mechanism pays an agent.

Heterogeneous capacities, multiple agents and items: Let c be the smallest agent capacity and assume it is the capacity of agent 1. Let agent 2 be any agent with capacity $> c$. There are $\geq c + 1$ items. It suffices to consider restricted valuation matrices v where $v_{ij} = 0$ when $i > 2$ or when $j > c + 1$ and $v_{ij} \equiv v_{i2}$ for $i = 1, 2$ and $2 \leq j \leq c + 1$. We partition these valuations into four sets A, B_1, B_1^+, B_2, as follows (we omit cases with ties and only define the assignment of items $1, \ldots, c + 1$):

- (A) $v_{21} > v_{11}$ and $v_{22} > v_{12}$. For these valuations in an optimal allocation agent 2 obtains the bundle $\{1, \ldots, c+1\}$.
- (B$_1$) $v_{11} > v_{21}$ and $v_{12} < v_{22}$. For these valuations in an optimal allocation items 1 is assigned to agent 1 and items $2, \ldots, c+1$ to agent 2.
- (B$_1^+$) $v_{11} - v_{21} > v_{12} - v_{22}$ and $v_{12} > v_{22}$. For these valuations in an optimal allocation items $1, \ldots, c$ are assigned to agent 1 and item $c+1$ is assigned to agent 2.
- (B$_2$) $v_{12} - v_{22} > \max\{0, v_{11} - v_{21}\}$. For these valuations in an optimal allocation item 1 is assigned to agent 2 and items $2, \ldots, c+1$ to agent 1.

Substituting the above in (18) we obtain that for $v \in B_1^+$, agent 1 does not envy agent 2 if and only if

$$h_1(v_2) - h_2(v_1) \le v_2(opt_2) - v_1(opt_2) = v_{22} - v_{12} .$$

Agent 2 does not envy agent 1 if and only if

$$\begin{aligned} h_2(v_1) - h_1(v_2) &\le v_1(opt_1) - v_2(opt_1) \\ &= v_{11} + (c-1)v_{12} - v_{21} - (c-1)v_{22} . \end{aligned}$$

Combining we obtain that there is no envy for $v \in B_1^+$, if and only if

$$v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \le h_1(v_2) - h_2(v_1) \le v_{22} - v_{12} . \tag{24}$$

For a fixed $\epsilon > 0$ and for $x > \epsilon$, the valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = v_{22} = 0$ is clearly in B_1^+. For such v the left hand side of (24) is

$$\begin{aligned} &v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \\ &= -(x+3\epsilon) - (c-1)(x+\epsilon) \\ &= -cx - (c+2)\epsilon \end{aligned}$$

Substituting in (24) we obtain

$$-cx - (c+2)\epsilon \le h_1(0,0) - h_2(x+3\epsilon, x+\epsilon) \le -(x+\epsilon) . \tag{25}$$

The valuation v such that $v_{11} = x + 3\epsilon$, $v_{12} = x + \epsilon$, $v_{21} = x + \epsilon$, and $v_{22} = x$ is also clearly in B_1^+. For such v the left hand side of (24) is

$$\begin{aligned} &v_{21} + (c-1)v_{22} - v_{11} - (c-1)v_{12} \\ &= x + \epsilon + (c-1)x - (x+3\epsilon) - (c-1)(x+\epsilon) \\ &= -(c+1)\epsilon \end{aligned}$$

From (24) we obtain

$$-(c+1)\epsilon \le h_1(x+\epsilon, x) - h_2(x+3\epsilon, x+\epsilon) \le -\epsilon. \tag{26}$$

Combining (25) and (26) we obtain,

$$\begin{aligned} h_1(x+\epsilon, x) &\le h_2(x+3\epsilon, x+\epsilon) - \epsilon \\ &\le h_1(0,0) + cx + (c+2)\epsilon - \epsilon \\ &= h_1(0,0) + cx + (c+1)\epsilon \end{aligned}$$

For valuations $v_{21} = x + \epsilon$, $v_{22} = x$, $v_{11} = v_{12} = x - \epsilon$, we clearly have $v \in A$ (agent 2 gets all items), hence $v_2(opt_2) = (c+1)x - (c+1)\epsilon$.

For a sufficiently large x (relative to ϵ and $h_1(0,0)$), $h_1(v_2) = h_1(x+\epsilon, x) \le h_1(0,0) + cx + (c+1)\epsilon < (c+1)x - (c+1)\epsilon = v_2(opt_2)$, which contradicts the no positive transfers requirement (23).

5 2 agents, Public Capacities

In this section we assume that capacities are public and derive $IC \cap EF$ payments for any game with two players.

Lemma 5.1. *Any 2-player capacitated allocation game with public capacities has an $IC \cap EF$ individually rational mechanism.*

Proof. Let c_i be the capacity of player i and assume without loss of generality that $c_1 \leq c_2$. For a vector $(x_1, x_2 \ldots)$ let $top_b\{x\}$ be the set of the b largest entries in x. We show that

$$h_1(v_2) = \sum_{j \in top_{c_1}\{v_2\}} v_{2j}$$

and

$$h_2(v_1) = \sum_{j \in top_{c_1}\{v_1\}} v_{1j}$$

give VCG payments which are envy-free.

It suffices to show that for $\{i, j\} = \{1, 2\}$,

$$h_i(v^{-i}) - h_j(v^{-j}) \leq v_j(opt_j) - v_i(opt_j) \ .$$

That is,

$$\sum_{j \in top_{c_1}\{v_2\}} v_{2j} - \sum_{j \in top_{c_1}\{v_1\}} v_{1j} \leq v_2(opt_2) - v_1(opt_2) \tag{27}$$

and

$$\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \leq v_1(opt_1) - v_2(opt_1). \tag{28}$$

Assume first that the number of items is exactly $c_1 + c_2$. In the optimal solution, player 1 will get the c_1 items that maximize $v_{1j} - v_{2j}$ and player 2 will get the c_2 items that minimize this difference.

We establish (28) as follows

$$\begin{aligned}
&\sum_{j \in top_{c_1}\{v_1\}} v_{1j} - \sum_{j \in top_{c_1}\{v_2\}} v_{2j} \\
\leq \quad & \sum_{j \in top_{c_1}\{v_1\}} (v_{1j} - v_{2j}) \\
\leq \quad & \sum_{j \in top_{c_1}\{v_1 - v_2\}} (v_{1j} - v_{2j}) \\
= \quad & \sum_{j \in opt_1} (v_{1j} - v_{2j}) = v_1(opt_1) - v_2(opt_1) \ .
\end{aligned}$$

We establish (27) as follows

$$\begin{aligned}
&\sum_{j\in top_{c_1}\{v_2\}} v_{2j} - \sum_{j\in top_{c_1}\{v_1\}} v_{1j} \\
&\leq \sum_{j\in top_{c_1}\{v_2\}} (v_{2j} - v_{1j}) \\
&\leq \sum_{j\in top_{c_1}\{v_2-v_1\}} (v_{2j} - v_{1j}) \\
&\leq \sum_{j\in opt_2} v_{2j} - \sum_{j\in top_{c_1}\{v_1(opt_2)\}} v_{1j}
\end{aligned}$$

where $v_1(opt_2)$ is the vector of the values of player 1 to the items player 2 gets in the optimal solution.

If there are fewer than $c_1 + c_2$ items, we add "dummy" items with valuations $v_{1j} = v_{2j} = 0$ and the lemma follows from the previous argument for the case with $c_1 + c_2$ items.

If there are more than $c_1 + c_2$ items then consider the set of $c_1 + c_2$ items than participate in the optimal solution. We now observe that (27) and (28) only involve items that participate in the optimal solution ($top_{c_2}\{v_2\}$ and $top_{c_1}\{v_1\}$ must both be included in the optimal solution). □

6 2 agents, 2 items, Private Capacities

In this section, valuations and capacities are private. We give VCG payments which are envy-free and individually rational for any game with two agents and two items. We specify the payments by giving the functions $h_1(v_2, c_2)$ and $h_2(v_1, c_1)$. Note that with two items, all $c_i \geq 2$ are equivalent, therefore we only need to consider capacities $\in \{0, 1, 2\}$.

We show that the following give envy-free payments

$$\begin{aligned}
h_1(v_2, c_2) &= \begin{cases} \max(v_{21}, v_{22}) & c_2 \in \{1,2\} \\ 0 & c_2 = 0 \end{cases} \\
h_2(v_1, c_1) &= \begin{cases} \max(v_{11}, v_{12}) & c_1 \in \{1,2\} \\ 0 & c_1 = 0 \end{cases}
\end{aligned}$$

The payments are envy-free if and only if

$$\begin{aligned}
\delta_{12} = h_1(v_2, c_2) - h_2(v_1, c_1) &\leq v_2(opt_2) - v_1(opt_2), \\
\delta_{21} = h_2(v_1, c_1) - h_1(v_2, c_2) &\leq v_1(opt_1) - v_2(opt_1).
\end{aligned}$$

The conditions when $\{c_1, c_2\} = \{1, 2\}$ were worked out in the previous section and the correctness for $h_1(v_2, 2)$ and $h_2(v_1, 1)$ carries over (and symmetrically, if we switch capacities of the agents). Consider the following remaining cases.

- $c_1 = c_2 = 2$**:** agent 1 does not envy agent 2 if and only if:

$$h_1(v_2, 2) - h_2(v_1, 2) \leq \begin{cases} v_{21} + v_{22} - v_{11} - v_{12} & \text{if } v_{21} > v_{11}, v_{22} > v_{12} \\ v_{22} - v_{12} & \text{if } v_{21} < v_{11}, v_{22} > v_{12} \\ v_{21} - v_{11} & \text{if } v_{21} > v_{11}, v_{22} < v_{12} \\ 0 & \text{if } v_{21} < v_{11}, v_{22} < v_{12} \end{cases}$$

Symmetrically, agent 2 does not envy agent 1 if and only if:

$$
\begin{aligned}
&h_2(v_1, 2) - h_1(v_2, 2) \leq \\
&\quad \begin{cases}
v_{11} + v_{12} - v_{21} - v_{22} & \text{if } v_{11} > v_{21}, v_{12} > v_{22} \\
v_{12} - v_{22} & \text{if } v_{11} < v_{21}, v_{12} > v_{22} \\
v_{11} - v_{21} & \text{if } v_{11} > v_{21}, v_{12} < v_{22} \\
0 & \text{if } v_{11} < v_{21}, v_{12} < v_{22}
\end{cases}
\end{aligned}
$$

Combining, we obtain the condition

$$
\begin{aligned}
&\min\{v_{21} - v_{11}, 0\} + \min\{v_{22} - v_{12}, 0\} \\
&\quad \leq \quad h_1(v_2, 2) - h_2(v_1, 2) \\
&\quad \leq \quad \max\{v_{21} - v_{11}, 0\} + \max\{v_{22} - v_{12}, 0\}.
\end{aligned} \tag{29}
$$

We now show that our particular h's satisfy (29). It suffices to establish one of the inequalities: We have

$$
\begin{aligned}
v_{21} &\leq \max\{v_{11}, v_{12}\} + \max\{v_{21} - v_{11}, 0\} \\
v_{22} &\leq \max\{v_{11}, v_{12}\} + \max\{v_{22} - v_{12}, 0\}
\end{aligned}
$$

Combining, we obtain the desired relation:

$$
\begin{aligned}
&\max\{v_{21}, v_{22}\} \\
&\quad \leq \quad \max\{v_{11}, v_{12}\} + \max\{v_{21} - v_{11}, 0\} + \max\{v_{22} - v_{12}, 0\}\ .
\end{aligned}
$$

• $c_1 = c_2 = 1$**:** agent 1 does not envy agent 2 if and only if:

$$
\begin{aligned}
&h_1(v_2, 1) - h_2(v_1, 1) \\
&\quad \leq \quad \begin{cases}
v_{22} - v_{12} & v_{11} + v_{22} > v_{12} + v_{21} \\
v_{21} - v_{11} & v_{11} + v_{22} < v_{12} + v_{21}
\end{cases}
\end{aligned}
$$

Symmetrically, agent 2 does not envy agent 1 if and only if:

$$
\begin{aligned}
&h_2(v_1, 1) - h_1(v_2, 1) \\
&\quad \leq \quad \begin{cases}
v_{12} - v_{22} & v_{21} + v_{12} > v_{22} + v_{11} \\
v_{11} - v_{21} & v_{21} + v_{12} < v_{22} + v_{11}
\end{cases}
\end{aligned}
$$

Combining, we obtain

$$
\begin{aligned}
&\min\{v_{22} - v_{12}, v_{21} - v_{11}\} \\
&\quad \leq \quad h_1(v_2, 1) - h_2(v_1, 1) \\
&\quad \leq \quad \max\{v_{22} - v_{12}, v_{21} - v_{11}\}
\end{aligned} \tag{30}
$$

We now show that our particular h's satisfy (30). It suffices to establish one of the inequalities: We have

$$
\begin{aligned}
v_{21} &\leq \max\{v_{11}, v_{12}\} + v_{21} - v_{11} \\
v_{22} &\leq \max\{v_{11}, v_{12}\} + v_{22} - v_{12}\}
\end{aligned}
$$

Combining, we obtain the desired relation:

$$\max\{v_{21}, v_{22}\} \leq \max\{v_{11}, v_{12}\} + \max\{v_{21} - v_{11}, v_{22} - v_{12}\} \ .$$

• $c_1 = 1, c_2 = 0$**:** No agent envies the other if and only if

$$\begin{aligned} h_1(v_2, 0) - h_2(v_1, 1) &\leq 0 \\ h_2(v_1, 1) - h_1(v_2, 0) &\leq \max\{v_{11}, v_{12}\} \end{aligned}$$

Combining, we obtain

$$-\max\{v_{11}, v_{12}\} \leq h_1(v_2, 0) - h_2(v_1, 1) \leq 0 \tag{31}$$

Symmetrically, when $c_1 = 0, c_2 = 1$:

$$-\max\{v_{21}, v_{22}\} \leq h_2(v_1, 0) - h_1(v_2, 1) \leq 0 \tag{32}$$

Our particular h's trivially satisfy (31) and (32).

• $c_1 = 2, c_2 = 0$**:** No agent envies the other if and only if

$$\begin{aligned} h_1(v_2, 0) - h_2(v_1, 2) &\leq 0 \\ h_2(v_1, 2) - h_1(v_2, 0) &\leq v_{11} + v_{12} \end{aligned}$$

Combining, we obtain

$$-v_{11} - v_{12} \leq h_1(v_2, 0) - h_2(v_1, 2) \leq 0 \tag{33}$$

Symmetrically, when $c_1 = 0, c_2 = 2$:

$$-v_{21} - v_{22} \leq h_2(v_1, 0) - h_1(v_2, 2) \leq 0 \tag{34}$$

Our particular h's trivially satisfy (33) and (34).

7 Conclusion and open problems

We have begun to study truthful and envy free mechanisms for maximizing social welfare for the capacitated allocation problem.

There is much left open, for example:

1. Is there a truthful and envy free mechanism (with positive transfers) for the capacitated allocation problem (arbitrary capacities):
   (a) With public capacities and more than two agents.
   (b) With private capacities for more than 2 agents and 2 items?
2. How well can we approximate the social welfare by a mechanism that is incentive-compatible, envy-free, invidually rational, and without positive transfers for capacitated allocations ?
3. Noam Nisan has observed that for superadditive valuations, there may be no mechanism that is both truthful and envy free. We conjecture that one can obtain mechanisms that are both truthful and envy free for subadditive valuations.

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